# Adaptive delayed feedback control algorithm with a state dependent delay

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*Summary*. We propose an adaptive modification of the delayed feedback control (DFC) algorithm that automatically finds the period of a target unstable periodic orbit (UPO) embedded in a chaotic attractor. The method is based on three dynamic equations that are added to the DFC controlled chaotic system. These equations can be experimentally implemented by low- and high-pass filters (LPF/HPF). They provide an asymptotic convergence of the variable delay time to the period of the target orbit.

## Introduction

Delayed feedback control (DFC) algorithm has been invented in the early 1990s [1] as a simple, robust, and efficient method to stabilize unstable periodic orbits (UPOs) in chaotic systems. Nowadays it has become one of the most popular methods in chaos control research [2]. The DFC algorithm is reference-free and makes use of a control signal obtained from the difference between the current state of the system and the state of the system delayed by one-period of a target orbit. The method allows a noninvasive stabilization of UPOs in the sense that the control force vanishes when the target state is reached. The controlled system can be treated as a black box, since the method does not require any exact knowledge of either the form of the periodic orbit or the system's equations. Successful implementations of the method include quite diverse experimental systems from different fields of science. For the details of experimental implementations as well as various modifications of the DFC algorithm we refer to the recent review paper [3].
The aim of this research is to construct an adaptive algorithm for finding the period of the target orbit. An adaptive discreet-time technique based on the gradient-descent method has been suggested in Refs. [4, 5]. Here we propose an adaptive continuous-time DFC algorithm with a state-dependent delay, which is based on resonance dependence of the DFC perturbation on the delay time pointed out in Ref. [1]. We implement an adaptive strategy similar to that proposed in Ref.[6] to maintain a synchronous state in time-evolving complex networks. Note that in the last five years the systems described by state dependent delay differential equations (SD DDEs) have become of particular interest. In Refs [7, 8], it was shown the beneficial effect of variation of the delay time on enhancing the stability properties of the DFC algorithm.

## The idea of the algorithm

Consider the DFC controlled dynamic system:

$$\dot{\boldsymbol{X}}(t) = \boldsymbol{F}[\boldsymbol{X}(t), K\Delta s(t)]. \tag{1}$$

The first argument in the function $\boldsymbol{F}$ shows the dependence of the vector field on internal degrees of freedom, while the second argument denotes the dependence on control force $K\Delta s(t)$. Here $K$ is the feedback gain, and the delayed difference is defined as $\Delta s(t) = s(t) - s(t-\tau) = g[\boldsymbol{X}(t)] - g[\boldsymbol{X}(t-\tau)]$. Here $s(t) = g[\boldsymbol{X}(t)]$ is a measurable scalar signal that is a function of all the variables. The free system ($K = 0$) has an unstable periodic solution $\boldsymbol{X}(t) = \boldsymbol{\xi}(t) = \boldsymbol{\xi}(t-T)$ that we intend to stabilize by control perturbation $K\Delta s(t)$ with time-dependent delay time $\tau(t)$. Our aim is to construct an algorithm, which provides an asymptotic convergence of the delay $\tau(t)$ towards the period of desired UPO and stabilizes this UPO.
As a helpful tool for this purpose, we construct the following functional:

$$\bar{\Delta}(t) = \int_0^t \exp[-\nu(t-t')][s(t') - s(t'-\tau(t'))]^2 dt'. \tag{2}$$

This functional describes the running time-average of the quantity $[s(t) - s(t-\tau(t))]^2$. It estimates the averaged value of this quantity in the time interval $[t-1/\nu, t]$. The parameter $\nu$ defines the window of averaging. Inside the integral, the delay is a function of $t'$, i.e. $\tau = \tau(t')$. However, we suppose that the delay time varies slowly inside of the interval $[t-1/\nu, t]$. Therefore we can use an approximation: $\tau(t') \to \tau(t)$, i.e. we may suppose that the delay does not vary inside of the averaging window. In that case we denote the functional as $\Delta(t)$.
If we chose $\tau$ as a constant of time, we could scan the values of $\tau$, and plot the dependence of the functional $\bar{\Delta}$ vs. $\tau$. At the periods of UPOs it is exactly zero, i.e. we have resonances for $\tau = T_k$ (here $T_k$ is the period of the $k$-th UPO).
We now determine the dynamics of the delay by the following equation:

$$\frac{d\tau}{dt} = -\beta \frac{d\Delta}{d\tau}. \tag{3}$$

We choose the value of the parameter $\beta$ in such a way as to satisfy $1/\beta >> 1/\nu$, i.e. the delay time $\tau$ evolves much more slower than the time of averaging. The form of Eq.(3) suggests that the functional $\Delta$ may be regarded as a potential in the space of delay $\tau$. Since the dependence $\Delta = \Delta(\tau)$ has a form of parabolas in the neighborhoods of periods of UPOs, one

may expect that the nearby solution of varied delay will converge towards the minima, $\tau \to T_k$, where $T_k$ is the period of the desired UPO.
In order to proceed, we exchange the argument of the delay, $\tau(t') \to \tau(t)$, thus obtaining

$$\bar{\Delta}(t) \approx \Delta(t) = \int_0^t \exp[-\nu(t-t')][s(t') - s(t'-\tau(t))]^2 dt'. \tag{4}$$

We define the new function $G(t) \equiv \frac{d\Delta}{d\tau}$. Differentiating Eq.(4), we get

$$G(t) \equiv \frac{d\Delta}{d\tau} = 2\int_0^t \exp[-\nu(t-t')][s(t') - s(t'-\tau(t))]\dot{s}(t'-\tau(t))dt'. \tag{5}$$

After differentiating $G(t)$ with respect to time (note that before this procedure we restore the argument of delay, i.e. we change back $\tau(t) \to \tau(t')$ in (5)), we can collect the above results into one system:

$$\dot{\boldsymbol{X}} = \boldsymbol{F}(\boldsymbol{X}, K[s(t) - s(t-\tau(t))]), \tag{6a}$$
$$\dot{\tau} = -\beta G, \tag{6b}$$
$$\dot{u} = \gamma[\boldsymbol{x}(t-\tau(t)) - u], \tag{6c}$$
$$\dot{G} = -\nu G + 2\gamma[s(t) - s(t-\tau(t))][s(t-\tau(t)) - u]. \tag{6d}$$

Eq.(6c) is used for approximate estimation of the time-derivative, i.e. we set $\dot{s}(t-\tau(t)) \to \gamma[s(t-\tau(t)) - u]$ in Eq.(6d). The Eqs.(6(b,c,d)) enable us to find the dynamics of the variable delay $\tau(t)$. The Eq.(6a) uses these dynamics in the DFC feedback perturbation in order to stabilize the desired UPO. The three last equations in (6) are written using some approximations. These approximations are valid if the parameters involved in these equations are properly chosen:

$$1/\gamma << 1/\nu << 1/\beta. \tag{7}$$

These inequalities impose the following conditions: (i) the dynamics of the delay $\tau$ must be much slower than that of the averaging time, and (ii) the dynamics of the additional HPF must be much faster than that of the averaging time. The first condition (i) enables us to assume that the delay $\tau$ is almost constant in the functional; therefore we may compute the gradient of the functional in the space of delay [see Eqs.(3), (4), (5)]. The second condition (ii) ensures the fast dynamics of the HPF, Eq.(6c), i.e. the filter follows the dynamics of delayed derivative.

## Results and conclusions

In the proposed algorithm, the search for the period of the target orbit is based on the gradient-descent method. The efficiency of proposed modification was confirmed by numerical simulations of adaptive DFC controlled Rössler and Mackey-Glass systems. The optimal controller parameters were found by computing Lyapunov exponents from the linearized equations of these systems. The target orbits can be stabilized by this modification even if the variable delay time starts relatively far away from the actual period of the desired UPO.
Note that Lin *et al.* [9] have recently proposed an adaptive controller with the delay time and control gain being both state-dependent. The controller provides a monotonous increase of the control gain, while the theoretical foundation of this algorithm is based on the assumption that trajectories of the controlled system are bounded. As a result the convergence of the delay time to the period of actual UPO has been attained only approximately. In contrast, our algorithm provides the exact convergence of the delay time to the period of unstable orbit and we do not need any additional implements like impulsive switching of control perturbation or truncation of the control force used in Ref. [9]. Our adaptive algorithm stabilizes the controlled orbit without changing its profile. The linear stability of orbits under action of adaptive control force is proven by computation of Lyapunov exponents.

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