Search for optimal signals to suppress synchronization in oscillatory networks

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Summary. We analyse a possibility to suppress the synchronization in oscillatory network via an external time-dependent force without the use of a feedback. The force is presented by a harmonic signal with the frequency equal to the center of natural frequencies distribution of the network's oscillators and the amplitude modulated by a periodic function with the different frequency. We formulate a variational problem to look for an optimal profile of this periodic function. The order parameter is taken as a criterion of the synchronization. We require the minimal value of the mean order parameter provided the mean power of the external force is fixed. Using a network of globally coupled Kuramoto oscillators we show that a smooth modulation signal satisfying the variational problem with the above constraints does not effectively suppress the synchronization, while a periodic switch on and off of the external force (a non-smooth modulation function) is considerably more effective.

Introduction

Synchronization is a phenomenon when weakly interacting oscillators adjust their intrinsic frequencies. Investigation of synchronization in large populations of interacting oscillatory elements is an intensively developing branch of nonlinear science, relevant to many problems of physics, chemistry, and biology, in particular, to neuroscience. Synchronization of individual neurons is believed to play the crucial role in the emergence of pathological rhythmic brain activity in Parkinson's disease, essential tremor, and epilepsies. The development of techniques for suppression of the undesired neural synchrony constitutes a challenging problem important from the standpoint of clinical applications [1]. Here our aim is to investigate the possibility to suppress synchronization in oscillatory networks via an external time-dependent force without the use of the feedback.

Problem formulation

As a model of interacting oscillatory elements we consider the Kuramoto equations [2]:

$$\dot{\theta}_j = \omega_j + \frac{K}{N}\sum_{k=1}^{N}\sin(\theta_k - \theta_j) + A(t)\sin(\omega_0 t - \theta_j). \tag{1}$$

Here θ_j and ω_j are respectively the phase and natural frequency of the j'th oscillator with $j = 1, \ldots, N$, where N is the number of oscillators in the network. We assume that the frequencies ω_j are randomly chosen from a Lorentzian probability density

$$g(\omega) = \frac{\Delta}{\pi[\Delta^2 + (\omega - \omega_0)^2]}, \tag{2}$$

where ω_0 is the center and Δ is the width of the distribution. The oscillators are globally coupled by the mean field with the strength K. The last term in Eq. (1) represents a harmonic force applied to all oscillators with the mean frequency ω_0 and periodically modulated amplitude $A(t)$. We use the standard synchronization criterion defined by the complex order parameter

$$r(t) = \frac{1}{N}\sum_{j=1}^{N}\exp\left(\imath\theta_j(t)\right). \tag{3}$$

The absolute value $|r|$ of this parameter varies in the interval $[0, 1]$ and measures the phase coherence in the population. The small values of $|r|$ indicate the incoherent state while values close to 1 represent the strong mutual synchronization. In the limit $N \to +\infty$, the order parameter satisfies the ordinary differential equation of the form (here we use the rotating coordinate system, i.e. replace $r(t)$ by $r(t)\exp(\imath\omega_0 t)$) [3]

$$\dot{r} = -\frac{1}{2}r^2\left(Kr^* + A(t)\right) + \frac{1}{2}\left(Kr + A(t)\right) - r\Delta, \tag{4}$$

where r^* means the complex conjugate of r. This equation can be rewritten into the rectangular coordinate system $r = x + \imath y$ from where we obtain that $y = 0$ is a steady state solution independently of $A(t)$ profile. The real part of the order parameter x has no steady state solution. However, if we chose $A(t)$ periodic with a period T then the real part of Eq. (4) has a periodic solution with the same period T: $x(t) = x(t + T)$. We say that solution $(x(t), 0)$ is stable when its infinitesimal perturbations $(\delta x(t), \delta y(t))$ vanishes for $t \to \infty$. Therefore, in order to reduce the synchronization level in the system we need to minimize $|x|$ variation through the period T taking into account the solution's stability requirement. Also we need to add a restriction to the amplitude of the external force $\int_0^T A^2(t)dt = W = const$, otherwise the minimization problem will give us the force with an infinite power. Mathematically, the problem can be stated as a minimization of the functional

$$J[x] = \int_0^T x^2(t)\mathrm{d}t, \tag{5}$$

with the above mentioned constraints.

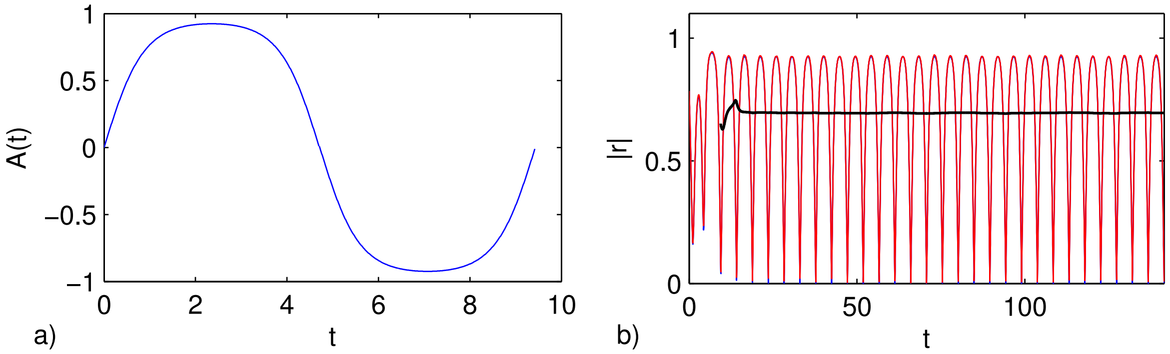

Figure 1: (a) Amplitude modulation signal obtained from solution of the Euler-Lagrange equations. (b) Comparison of the absolute value of order parameter obtained from Eqs. (1) (red curve) and Eqs. (4) (blue curve) with $A(t)$ shown in panel (a). Black curve represents a moving average of the order parameter with averaging period equal to $T = 9.4$. The value of the parameters are: $N = 10000$, $K = 0.5$, $\omega_0 = 0.1\pi$, and $\Delta = 0.1$.

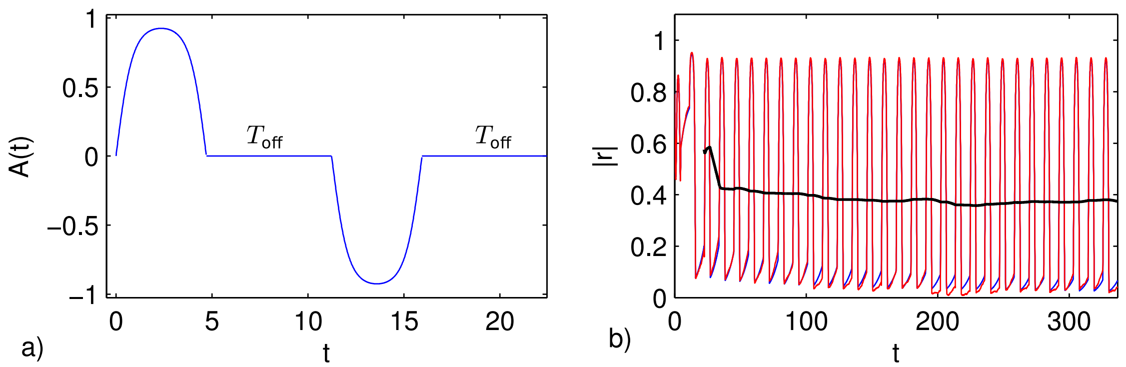

Figure 2: The same as in Fig. 1, but now the external force is turned off $A(t) = 0$ for intervals $T_{\text{off}} = 6.5$.

Results

The solution of the variational problem by the Euler-Lagrange method leads to the system of two nonlinear ordinary differential equations for $x(t)$ and $A(t)$ functions. Solving this system with the periodicity conditions we obtain a family of curves A(t) corresponding to the different values of the mean power W/T of the force. Comparing these solutions we can select the solution with the minimal power. Panel (a) of Fig. 1 shows an amplitude modulation signal $A(t)$ obtained from the solution of the Euler-Lagrange equations, while panel (b) demonstrates the dynamics of the order parameter with the given periodic signal $A(t)$. The dynamics of the order parameter obtained directly from the Kuramoto model (1) (red curve) and Eqs. (4) (blue curve) practically coincide. We see that the order parameter vanishes two times per period T, i.e. the external force totally desynchronize the the system two times per period. However, the mean value of the order parameter remains rather large $\langle |r| \rangle \approx 0.7$; it is comparable with the order parameter of the free system $|r| \approx 0.77$ for the given value of $K = 0.5$. To reduce the mean value of the order parameter we can switch off the external force at the moments when the order parameter is close to zero. However, the free system does not remain in the desynchronized state a long while, since the fixed point $r = 0$ of the free system (4) is unstable. Nevertheless, we can switch off the external force for some interval T_{off} and than switch it on again. The allowable length of T_{off} interval is restricted by the stability condition of the given solution. The non-smooth periodic signal $A(t)$ with zero values in T_{off} intervals is shown in panel (a) of Fig. 2. Panel (b) shows the results of application of this signal to the Kuramoto system. Now the averaged value of the order parameter $\langle |r| \rangle \approx 0.4$ is considerably reduced in comparison to the corresponding value of the free system. Therefore the non-smooth intermittent signals are more effective for the suppression of synchronization.

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