The anticipating of chaotic dynamics via an act-and-wait control

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Summary. An act-and-wait algorithm is proposed to design the coupling law for anticipating synchronization of chaotic systems. The time-delay coupling term in this algorithm is periodically switched on and off such that the phase space of the whole system remains finite-dimensional despite the presence of a delay. Consequently, the stability of a response system is easily to attain, since it is definite by the finite number of Lyapunov exponents. We show that the stable synchronization regime with considerably large anticipation time can be achieved even when the single output of the response and the single input for the drive systems is used. The results are demonstrated with the Rössler and Chua systems.

Introduction

An anticipating synchronization (AS) was introduced for the forecasting of chaotic dynamics in Ref. [1]. It deals with a particular regime appearing in unidirectionally coupled systems in a drive - response (master - slave) configuration

$$\dot{\mathbf{r}}_1(t) = \mathbf{f}(\mathbf{r}_1(t)), \quad (1a)$$
$$\dot{\mathbf{r}}_2(t) = \mathbf{f}(\mathbf{r}_2(t)) - \mathbf{K}\left(\mathbf{r}_2(t-\tau) - \mathbf{r}_1(t)\right), \quad (1b)$$

as two dynamical systems $\mathbf{r}_1$ and $\mathbf{r}_2$ synchronize in such a way that a response system anticipates the trajectory of a drive system, $\mathbf{r}_2(t) = \mathbf{r}_1(t+\tau)$. In the original paper Ref. [1], a coupling matrix $\mathbf{K}$ has been chosen in the form of a constant diagonal matrix, and a stable synchronization regime was achieved only for the small value of an anticipation time τ as compared with the characteristic period of chaotic oscillations. To enlarge the anticipation time, more sophisticated algorithms for designing the coupling matrix $\mathbf{K}$ were proposed in [2, 3]. However, for any constant matrix $\mathbf{K}$ the problem of the stability of an anticipation manifold is rather complicated due to the infinite number of Lypunov exponents caused by a time-delayed feedback term in the response system. To overcome this problem, here we propose to use an act-and-wait concept [4]. The point of the method is that the coupling term with time-delayed feedback is periodically switched on (act) and off (wait). If the duration of waiting (when the coupling is switched off) is larger than the feedback delay, then the problem of the stability of the anticipation manifold facilitates; it is characterized by the finite number of Lypunov exponents.

Results

Our algorithm is designed for the Rössler-type systems, whose chaotic attractors are originated from a saddle-focus fixed point. Specifically, the Rössler system, defined by a vector field

$$\mathbf{f}(\mathbf{r}) = [-y-z, x+ay, b+z(x-c)], \ \mathbf{r} = (x, y, z) \quad (2)$$

with $a = 0.15$, $b = 0.2$, and $c = 10$, has a saddle-focus fixed point close the origin with an unstable 2D spiral manifold that almost coincide with the (x, y) plane and a stable $1D$ manifold that almost coincides with the z axes. Since the system spends most time in the (x, y) plane spiraling out from the fixed point we can take $z = 0$ and consider a two-dimensional vector field $\mathbf{f}(\mathbf{r}) = [-y, x+ay]$, $\mathbf{r} = (x, y)$ as a simplified model for the verification of various schemes of the AS [2]. Based on this simplified model we have devised the following coupling scheme:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{y}_1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & a \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \ \begin{pmatrix} \dot{x}_2 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & a \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} k(t)F(t). \quad (3)$$

This model describes the AS between two unstable spirals with an increment $\gamma = a/2$ and a frequency $\omega = \sqrt{1-\gamma^2}$. The single output and the single input coupling of a form

$$F(t) = x_2(t-\tau) - x_1(t) + p[x_2(t-\tau-\theta) - x_1(t-\theta)] \quad (4)$$

is applied only to the first equation of the response system. Contrary to usual AS schemes, which use a single delay, here we introduce a second delay, where $\theta < \tau$ and p defines the relative strength of a second coupling term. The time-dependent parameter $k(t)$ realizes the act and wait strategy. Its values change periodically with the period 2τ from $k = 0$ (wait) to $k = k_0$ (act). More specifically, in the intervals $t_j < t \leq t_j + \tau + \theta$ the parameter $k = 0$, while in the intervals $t_j + \tau + \theta < t \leq t_j + 2\tau$ the value of this parameter is $k = k_0 = const$. Here $t_j = 2j\tau$, $j = 0, 1, 2, \ldots$ are the points at which the coupling is switched off. The duration of an act interval $\tau_a = \tau - \theta$ is less that the duration of a wait interval $\tau_w = \tau + \theta$. It is easy to see that the anticipatory synchronization manifold $\mathbf{r}_2(t) = \mathbf{r}_1(t+\tau)$ is the solution of Eqs. (3). The stability of this solution is defined by the variational equation for a deviation $\delta\mathbf{r}(t) = \mathbf{r}_2(t) - \mathbf{r}_1(t+\tau)$:

$$\begin{pmatrix} \delta\dot{x} \\ \delta\dot{y} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & a \end{pmatrix} \begin{pmatrix} \delta x \\ \delta y \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} k(t)[\delta x(t-\tau) + p\delta x(t-\tau-\theta)]. \quad (5)$$

Though these equations contain time-delay terms, they are periodically switched on and off, and this allows us to solve (5) as the system of ordinary differential equations. Taking $\delta\mathbf{r}(t_j) = \delta\mathbf{r}_j \equiv (\delta x_j, \delta y_j)$ as an initial condition and solving the system (5) in the periodicity interval of $k(t)$: $t_j \leq t \leq t_{j+1} = t_j + 2\tau$, we obtain the relationship between $\delta\mathbf{r}_{j+1}$ and $\delta\mathbf{r}_j$ in a simple matrix form:

$$\delta\mathbf{r}_{j+1} = \mathbf{M}\delta\mathbf{r}_j, \tag{6}$$

where $\mathbf{M}$ is a constant 2×2 monodromy matrix. The AS state is stable if the both eigenvalues of this matrix are inside of the unity circle of the complex plane. Thus the stability of the AS regime is defined by only two eigenvalues despite the presence of time-delay terms. For a fixed anticipation time τ the eigenvalues of the monodromy matrix are controlled by the coupling parameters p, k_0, and θ. Using analytical arguments we found that a good choice for the parameter p is as follows:

$$p = -\frac{\sin(\omega\tau)}{\sin(\omega\tau_w)} \exp(\gamma\theta). \tag{7}$$

This expression holds for any Rössler-type systems. Numerical analysis shows that by appropriate choice of the coupling parameters the both eigenvalues of the monodromy matrix can be placed into unite circle of the complex plane for arbitrary large anticipation time τ. Thus the algorithm is indeed efficient for the simple linear system (3).

Application of the above algorithm to the original nonlinear Rössler system (2) is successful as well. Now the linear stability analysis leads to Eq. (6) with the 3D vector $\delta\mathbf{r}_j$ and the 3×3 monodromy matrix that depends on the index j: $\mathbf{M} = \mathbf{M}_j$. The matrices $\mathbf{M}_j$ produce three transversal Lyapynov exponents that define the stability of the anticipatory manifold. Again, the stability is defined by a finite number of Lyapunov exponents despite the presence of the delay terms. Our algorithm allowed us to achieve the stable AS regime in the Rössler system for sufficiently large anticipation time, which is comparable with that obtained by the phase-lag compensation coupling (PLCC) method introduced in Ref. [2]. For example, the stable AS with the prediction time $\tau = 3$ has been attained using the following values of the parameters: $\theta = 0.1$ and $k_0 = 0.66$. The parameter $p = -2.971$ was computed from Eq. (7) taken the frequency $\omega = 0.977$ and the increment $\gamma = 0.074$ of the spiral derived from the eigenvalues of the fixed point of the Rössler system. Note, that this algorithm is considerably simpler than the PLCC algorithm from the viewpoint of feedback in both the theoretical analysis and the practical implementation.

The act-and-wait control method is also suitable for the online forecasting the chaotic dynamics in the double-scroll Chua system. The vector field of the Chua circuit has a form:

$$\mathbf{f}(\mathbf{r}) = \left(\left[g(y-x) - i_N\right]/C_1, \left[g(x-y) + z\right]/C_2, -y/L\right), \; \mathbf{r} = (x, y, z). \tag{8}$$

Here $i_N = m_0 x + 0.5(m_1 - m_0)(|x + B_p| - |x - B_p|)$ is a piecewise-linear function. We choose the parameters $C_1 = 0.1,\ C_2 = 1,\ L = 1/2,\ m_0 = -0.5,\ m_1 = -0.8,\ B_p = 1$ and $g = 0.58$ such that the circuit operates in the double-scroll regime. The double-scroll attractor originates from two saddle-focus fixed points with identical eigenvalues. The parameters of the unstable 2D spiral manifold are: $\gamma = 0.117$ and $\omega = 1.5698$. Unlike to the Rössler system, here the coupling term has been added to the second equation using y as an output variable. Again, the maximal anticipation time attained with our algorithm is comparable with that obtained with the PLCC algorithm in Ref. [2]. For example, the stable AS with the prediction time $\tau = 2.5$ has been attained by taking $\theta = 0.2$, $k_0 = 0.95$, and $p = -0.8115$. The latter parameter has been again computed from Eq. (7).

Conclusions

The act-and-wait concept is an efficient tool to design the coupling law for the anticipating synchronization of chaotic systems. By periodical switching on and off the coupling term we can reduce the infinite dimensional phase space of a time-delay system to the finite dimensional phase space. As a result the stability of such systems is easily to achieve, since we need to control only the finite number of Luapunov exponents. We have shown that the proposed algorithm works well for the Rössler and Chua systems and provides a long-term anticipation. In comparison to a previously proposed PLCC method [2], this algorithm is easily to implement, since contrary to [2] it is applicable to the systems with a single output and a single input.

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