Time-delayed feedback control of periodic orbits with an odd-number of positive unstable Floquet multiplieres

Viktor Novičenko* and Kestutis Pyragas *
*Center for Physical Sciences and Technology, Semiconductor Physics Institute, Vilnius, Lithuania

Summary. We present a control matrix design algorithm for stabilization of unstable periodic orbits in autonomous systems using the time-delayed feedback control, when the orbits have an odd number of real Floquet multipliers larger than unity. Due to the so-called odd-number theorem such orbits have been considered as uncontrollable by time-delayed feedback methods. However, the theorem has been refuted for autonomous systems by a counterexample and recently a corrected version of the odd number theorem has been proved. Our algorithm is based on relationship between Floquet multipliers of the system controlled by time-delayed and proportional feedback. The efficiency of our approach is demonstrated with the Lorenz system.

Motivation

Stabilization of unstable periodic orbits (UPOs) using a time-delayed feedback control (TDFC) is an important issue in control theory. The TDFC was successfully implemented in quite diverse experimental systems from different fields of science. However, Nakajima [1] formulated a theorem, which states that unstable periodic orbits with an odd number of real Floquet multipliers (FMs) larger than unity cannot be stabilized by TDFC. The theorem seemed to be supported by experimental and numerical evidence for both autonomous and nonautonomous systems, but ten years after Nakajima's publication Fiedler et al. [2] have shown by a counterexample that the theorem is false for autonomous systems (for non-autonomous systems the theorem is valid). Fiedler et al. considered an equation for a normal form of subcritical Hopf bifurcation that has an UPO with one unstable FM and showed that it can be stabilized by the TDFC. Later on different authors demostrated that the control matrix devised by Fiedler et al. works close to a subcritical Hopf bifurcation in other examples, but no examples were considered far away from a bifurcation point. Here our aim is to present a practical recipe for a design of the control matrix when the UPO is far away from a bifurcation point.

Control matrix design algorithm: Application for the Lorenz system

Let's say we have an n-dimensional dynamical system $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t))$ with an UPO $\mathbf{x}(t) = \mathbf{p}(t) = \mathbf{p}(t+T)$, which we seek to stabilize using TDFC:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)) + \mathbf{K}[\mathbf{x}(t-\tau) - \mathbf{x}(t)], \tag{1}$$

where $\mathbf{K}$ is a constant control matrix. If delay time τ is equal to the UPO period T then the UPO $\mathbf{p}(t)$ of the free system is also the solution of the controlled system (1) for any $\mathbf{K}$. But if the delay time τ is close but not equal to T then in the system (1) exists a periodic solution which is close to $\mathbf{p}(t)$. This mismatched periodic solution has a new period Θ, which depends on τ and $\mathbf{K}$. Recently Hooton and Amann [3] proved a corrected version of the odd number theorem for autonomous systems, which states that $\mathbf{p}(t)$ is an unstable periodic solution of the controlled system (1) for $\tau = T$ if the condition

$$(-1)^m \lim_{\tau \to T} \frac{\tau - T}{\tau - \Theta(\mathbf{K}, \tau)} < 0 \tag{2}$$

holds. Here m is a number of real FMs larger than unity for the periodic solution $\mathbf{p}(t)$ of the free system. We see that if m is odd then the stabilization of the periodic solution $\mathbf{p}(t)$ is possible only if the factor $\beta = \lim_{\tau\to T}(\tau - T)/(\tau - \Theta)$ is negative. Therefore we need to expand Θ in terms of a small mismatch $\tau - T$ up to the second order. This was done in [4, 5] and it allows us to rewrite β in a more useful form:

$$\beta = 1 + \sum_{ij}^{n} K_{ij} C_{ij}. \tag{3}$$

Here K_{ij} is the i,j-th element of the control matrix and $C_{ij} = \int_0^T \rho_i(t)\dot{p}_j(t)dt$, where $\rho_i(t)$ is the i-th component of the phase response curve of the UPO and $\dot{p}_j(t)$ is a derivative of the j-th component of the UPO. Note that the coefficients C_{ij} in (3) are independent of the control parameters $\mathbf{K}$ and τ. Thus by appropriate choice of $\mathbf{K}$ the factor β can be made negative. However, this condition is not sufficient for the successful control. We introduce additional criteria [6] which we derive from relationship between the FMs of the proportional and time-delayed feedback control. First let us rewrite the control matrix in the form $\mathbf{K} = \kappa\tilde{\mathbf{K}}$ where κ is a scalar control gain and $\tilde{\mathbf{K}}$ is a matrix with at least one element equal to -1 or 1 and other elements in the interval $[-1, 1]$. For each $\tilde{\mathbf{K}}$ we can choose κ in such a way as to make $\beta < 0$. Without loss of the generality we assume that sum $\sum_{ij}^{n} \tilde{K}_{ij} C_{ij}$ is negative, since this can be always achieved by appropriate choice of the sign of the matrix $\tilde{\mathbf{K}}$. In order to overcome the condition (2) the control gain has to be higher than its threshold value:

$$\kappa > \kappa^* \equiv -\left(\sum_{ij}^{n} \tilde{K}_{ij} C_{ij}\right)^{-1}. \tag{4}$$

Now let us consider the system under proportional feedback control (PFC):

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)) + g\tilde{\mathbf{K}}[\mathbf{p}(t) - \mathbf{x}(t)]. \tag{5}$$

Here a real-valued scalar g is a gain of the PFC. The information about the real FMs in the TDFC system can be obtained from the FMs of the PFC system using the following relationship [6]:

$$\lambda = \Lambda(g), \quad \kappa = g/\left[1 - \exp\left(-\Lambda(g)T\right)\right]. \tag{6}$$

Here $\Lambda(g)$ is the Floquet exponent (FE) in the PFC system and λ is the FE in the TDFC system. If the UPO of the free system has one real FM larger than unity (case $m = 1$) then there are two important branches $\Lambda(g)$: trivial ($\Lambda_0(0) = 0$) and unstable ($\Lambda_u(0) > 0$). In the Lorenz system with the standard parameters $\mathbf{f}(\mathbf{x}) = [10(x_2 - x_1), x_1(28 - x_3) - x_2, x_1x_2 - 8/3x_3]^T$ the successful stabilization of the period-one UPO can be easily achieved for the PFC case by choosing only one nonzero element $\tilde{K}_{22} = 1$ of the matrix $\tilde{\mathbf{K}}$ (see Figure 1 (a)). However, transforming the two PFC branches to the TDFC case (see Figure 1 (b)) using (6) we reveal that the unstable branch "crosses" the zero only for $\kappa \to \infty$. The successful

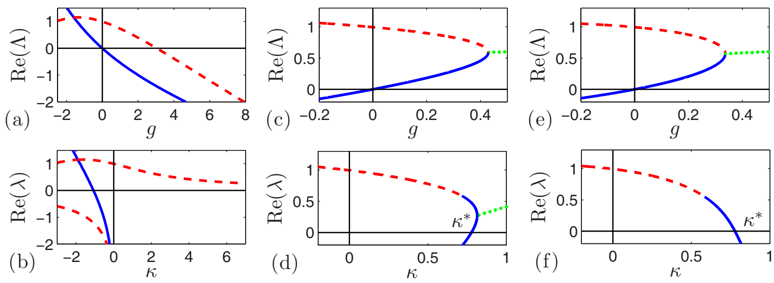

Figure 1: Three typical scenarios for the dependence of the FEs of the Lorenz system on the feedback gain for PFC (upper row) and TDFC (lower row) at different matrixes $\tilde{\mathbf{K}}$: (a) and (b) $[0, 0, 0; 0, 1, 0; 0, 0, 0]$; (c) and (d) $[0, 0, 0; -1, 0, 0.3; 0, 0, 0]$; (e) and (f) [0,0,0;-1,0,0.5;0,0,0]. In (a), (c) and (e), blue solid and red dashed curves represent trivial and unstable FEs (both real-valued) for PFC, respectively. In (b), (d) and (f), the corresponding curves show reconstructed values of FEs for TDFC. Green dotted curves in (c), (d) and (e) show real parts of complex conjugate FEs.

stabilization in the TDFC case can be achieved only if the trivial and the unstable branches in the PFC case coalesce (see Figure 1 (c) and (e)). In this scenario, the two branches in the PFC system transforms into one branch of the TDFC system (see Figure 1 (d) and (f)). However, only (f) case is potentially successful, since the branch $\lambda(\kappa)$ crosses the zero with the negative slope. The threshold κ^* of the control gain where the branch of the TDFC system crosses zero $\lambda(\kappa^*) = 0$ and the slope $\frac{d\lambda}{d\kappa}\big|_{\kappa=\kappa^*}$ at the intersection point can be computed from the Taylor expansion of the trivial branch $\Lambda_0(g)$ of the PFC system at the point $g = 0$:

$$\Lambda_0(g)T = ag + bg^2 + O(g^3). \tag{7}$$

By substituting (7) into (6) and taking the limit $g \to 0$ we get $\kappa^* = a$. This gives an alternative way to compute the threshold gain, since expansion coefficients a and b can be simply evaluated from the PFC system for any matrix $\tilde{\mathbf{K}}$. The condition for the negative slope $\frac{d\lambda}{d\kappa}\big|_{\kappa=\kappa^*} < 0$ can be again obtained from (7) and (6):

$$1 - 2b/a^2 < 0. \tag{8}$$

The successful stabilization of the period-one orbit of the Lorenz system is shown in Fig. 2.

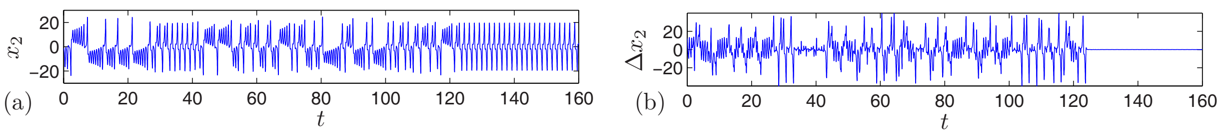

Figure 2: The dynamics of the successful stabilization of the period-one orbit in the Lorenz system. The matrix $\tilde{\mathbf{K}}$ is chosen the same as in Fig. 1 (e) and (f). (a) dynamics of the second variable. (b) dynamics of the difference $\Delta x_2(t) = x_2(t) - x_2(t - \tau)$. The control gain $\kappa = 0.865$.

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