Phase-reduction-theory-based treatment of extended delayed feedback control algorithm in the presence of a small time delay mismatch

Viktor Novičenko and Kestutis Pyragas
Center for Physical Sciences and Technology, A. Goštauto 11, Vilnius LT-01108, Lithuania
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The delayed feedback control (DFC) methods are noninvasive, which means that the control signal vanishes if the delay time is adjusted to be equal to the period of a target unstable periodic orbit (UPO). If the delay time differs slightly from the UPO period, a nonvanishing periodic control signal is observed. We derive an analytical expression for this period for a general class of multiple-input multiple-output systems controlled by an extended DFC algorithm. Our approach is based on the phase-reduction theory adapted to systems with time delay. The analytical results are supported by numerical simulations of the controlled Rössler system.

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I. INTRODUCTION

The use of time delayed signals for controlling unstable periodic orbits (UPOs) in complex dynamical systems has attracted considerable interest for the past two decades. A method introduced by Pyragas [1], known as delayed feedback control (DFC), does not require any exact model of controlled objects and complicated computer processing for the reconstruction of underlying dynamics. The control scheme consists of measuring an output signal $s(t)$, taking the time delayed difference $s(t) - s(t-\tau)$ amplified by some factor K, and using this control signal to modulate an input parameter of the system. By appropriate adjustment of the control amplitude K, the successful stabilization of a time-periodic state can be achieved. When the delay time τ is chosen equal to the period T of the target UPO, the control scheme is clearly noninvasive since the control force $K[s(t) - s(t-\tau)]$ vanishes when the target state is reached. An important modification of the DFC algorithm, known as an extended DFC (EDFC), has been introduced by Socolar *et al.* [2]; it uses multiple delays in the form of an infinite series and allows the stabilization of highly unstable orbits [3]. The DFC and EDFC algorithms have been applied successfully in diverse research areas. A review of experimental and theoretical achievements until 2006 can be found in Ref. [4]. The recent developments of the DFC algorithm include the refuting of the odd number limitation [5–9], the analysis of basins of attraction of the stabilized orbits [10–12], the DFC-based bifurcation analysis for experiments [13,14], the DFC with a time-varying delay [15], and adaptive modifications of the DFC [16,17]. Yamasue *et al.* [18] have recently demonstrated an important practical application of the DFC in an atomic force microscope to stabilize cantilever oscillation and to remove artifacts on a surface image.

Experimental implementation of the DFC method requires knowledge of the period T of a target UPO. For autonomous systems, this period is not known *a priori*, and a number of algorithms for estimating this period from an observed control signal has been developed. The algorithms are based on the fact that the amplitude of the control signal has a resonance-type dependence on the delay time [1], and the period of the UPO can be extracted from the minima of this dependence using various adaptive techniques [16,19–23]. A sophisticated theoretical foundation for obtaining the UPO period has been developed by Just *et al.* [24]. In the case of successful stabilization of a target UPO, the periodic oscillations conserve in the controlled system, even for $\tau \neq T$, provided the mismatch $\tau - T$ is small, however, the period Θ of these oscillations differs from the UPO period T. In Ref. [24], an analytical expression for the period Θ has been derived up to second order in the mismatch,

$$\Theta(K,\tau) = T + \frac{K}{K-\kappa}(\tau - T) + O[(\tau - T)^2]. \quad (1)$$

Here, κ is a system parameter which captures all the details concerning the coupling of the control force to the system. The theory presented in Ref. [24] as well as Eq. (1) is restricted to the case of the DFC algorithm applied to the systems having a single scalar input.

The main goal of this paper is to derive an analytical expression for the period Θ for the EDFC algorithm applied to a general class of multiple-input multiple-output (MIMO) systems. Our approach is based on the phase-reduction theory of weakly perturbed limit cycle oscillations in systems with time delays, which was developed in our recent paper [25]. We also present the algorithm for computation of the parameter κ and other inherent system parameters arising in our generalized expression for the period Θ. Moreover, the theoretical results obtained in this paper are applicable to a wider class of problems; they can be used to analyze the influence of any weak time-dependent perturbations on dynamics of the controlled system.

The rest of the paper is organized as follows. In Sec. II, we present a mathematical formulation of the problem. In Sec. III, we consider the EDFC algorithm for $\tau = T$ in the presence of a weak time-dependent perturbation and derive a phase-reduced equation. Section IV is devoted to the case $\tau \neq T$ and the derivation of an analytical expression for the period Θ. The obtained theoretical results are demonstrated numerically for the Rössler system in Sec. V. The paper is finished with the conclusions presented in Sec. VI.

II. PROBLEM FORMULATION

The EDFC algorithm has been originally introduced for a dynamical system with a scalar control variable [2]. In this paper, we consider the very general version of this algorithm applied to a MIMO system,

$$\dot{\mathbf{x}}(t) = \mathbf{f}[\mathbf{x}(t),\mathbf{u}(t)], \tag{2a}$$

$$\mathbf{s}(t) = \mathbf{g}[\mathbf{x}(t)], \tag{2b}$$

$$\mathbf{u}(t) = \mathbf{K}\left[(\mathbf{I}-\mathbf{R})\sum_{j=1}^{\infty}\mathbf{R}^{j-1}\mathbf{s}(t-j\tau)-\mathbf{s}(t)\right]. \tag{2c}$$

Here, $\mathbf{x}(t) \in \mathbb{R}^n$ denotes the state vector of the system, $\mathbf{u}(t) \in \mathbb{R}^k$ is a control vector variable (k-dimensional input), and $\mathbf{f}(\mathbf{x},\mathbf{u})$ is a nonlinear vector function that defines the dynamical laws of the system and the input properties of the control variable. Equation (2b) defines an l-dimensional output signal $\mathbf{s}(t)$, which is related to the n-dimensional state vector $\mathbf{x}$ through a vector function $\mathbf{g}: \mathbb{R}^n \to \mathbb{R}^l$. Equation (2c) gives the EDFC relation between the output vector variable $\mathbf{s}$ and the control vector variable $\mathbf{u}$. The diagonal $l \times l$ matrix $\mathbf{R} = \mathrm{diag}(R_1,\ldots,R_l)$ defines a set of memory parameters R_m, which generally can be different for the different components s_m of the output signal. However, we assume that the delay time τ for all components is the same. To provide the convergence of the infinite sum in Eq. (2c), we require that $|R_m| < 1$ for $m = 1,\ldots,l$. The control $k \times l$ matrix $\mathbf{K}$ defines the transformation of the output variable $\mathbf{s}(t)$ to the control (input) variable $\mathbf{u}(t)$, and $\mathbf{I}$ denotes an $l \times l$ identity matrix.

We suppose that the control-free ($\mathbf{K} = 0$) system has an unstable T-periodic orbit $\boldsymbol{\xi}(t) = \boldsymbol{\xi}(t+T)$ that satisfies the equation $\dot{\boldsymbol{\xi}}(t) = \mathbf{f}[\boldsymbol{\xi}(t),0]$. The aim of the EDFC signal (2c) is to stabilize this orbit. If we take the delay time equal to the UPO period $\tau = T$ and if the stabilization is successful, then $\mathbf{s}(t-j\tau) = \mathbf{s}(t)$, $\sum_{j=1}^{\infty}\mathbf{R}^{j-1} = (\mathbf{I}-\mathbf{R})^{-1}$, and the control variable vanishes $\mathbf{u} = 0$. This means that the control law (2c) applied to a MIMO system (2a) is noninvasive.

The aim of this paper is to analyze the situation when the delay time differs from the period of the UPO $\tau \neq T$. We assume that the control parameters are chosen such that, for $\tau = T$, the stabilization is successful, and the controlled system demonstrates stable periodic oscillations with the period $\Theta = T$. If we slightly detune the delay time τ, then the system still remains in the regime of stable periodic oscillations, but the period Θ of these oscillations is changed $\Theta \neq T$. We are seeking to derive an analytical expression for the period Θ in the dependence of the system parameters using the general formulation of the problem defined by Eqs. (2).

To solve this problem, we appeal to the phase-reduction theory of systems with multiple time delays, which is outlined in Appendix A. In the next section, we obtain the phase-reduced equations for a slightly perturbed system (2) with $\tau = T$ and show that the profile of the phase response curve (PRC) of controlled orbit does not depend on the control and memory matrices $\mathbf{K}$ and $\mathbf{R}$. Then, in Sec. IV, we consider the case $\tau \neq T$ and split the control force into mismatched and nonmismatched components. Treating the mismatched component as a small perturbation in the phase-reduction procedure, we derive an analytical expression for the period Θ.

III. PHASE REDUCTION OF THE SLIGHTLY PERTURBED EDFC SYSTEM FOR $\tau = T$

In this section, we consider the system (2) assuming that $\tau = T$ and the parameters of the control $\mathbf{K}$ and memory $\mathbf{R}$ matrices are chosen such that the stabilization of the target UPO is successful. Then, $\mathbf{x} = \boldsymbol{\xi}(t)$ is the stable periodic solution of the system (2). We are interested in how this solution will change in the presence of small perturbations. To this end, we add, to the right-hand side of Eq. (2a), a small perturbing term $\varepsilon\boldsymbol{\psi}(t)$ and taking into account that $\tau = T$, rewrite the system (2) as follows:

$$\dot{\mathbf{x}}(t) = \mathbf{f}[\mathbf{x}(t),\mathbf{u}_0(t)] + \varepsilon\boldsymbol{\psi}(t), \tag{3a}$$

$$\mathbf{s}(t) = \mathbf{g}[\mathbf{x}(t)], \tag{3b}$$

$$\mathbf{u}_0(t) = \mathbf{K}\left[(\mathbf{I}-\mathbf{R})\sum_{j=1}^{\infty}\mathbf{R}^{j-1}\mathbf{s}(t-jT)-\mathbf{s}(t)\right]. \tag{3c}$$

Here, $\boldsymbol{\psi}(t) = [\psi_1(t),\ldots,\psi_n(t)]^T$ is a time-dependent n-dimensional vector, and ε is a small parameter $|\varepsilon| \ll 1$. To distinguish the case $\tau = T$, we have marked the control variable by the zero subscript.

The sum in Eq. (3c) incorporates an infinite number of delays, and formally, the initial value for the EDFC is an infinite history. However, in reality, only a finite number of delays is practical since the influence of longer delays decreases exponentially. To avoid the problem with an infinite memory, we consider the EDFC with a finite number M of delay terms by replacing Eq. (3c) with

$$\tilde{\mathbf{u}}_0(t) = \mathbf{K}\left[\mathbf{P}\sum_{j=1}^{M}\mathbf{R}^{j-1}\mathbf{s}(t-jT)-\mathbf{s}(t)\right], \tag{4}$$

and after the derivation of the final result, we take the limit $M \to \infty$. Here, the matrix,

$$\mathbf{P} = (\mathbf{I}-\mathbf{R})(\mathbf{I}-\mathbf{R}^M)^{-1} \tag{5}$$

is introduced to ensure the noninvasiveness of the control scheme for any finite M since $\sum_{j=1}^{M}\mathbf{R}^{j-1} = \mathbf{P}^{-1}$. For $M \to \infty$, Eq. (4) transforms to Eq. (3c).

The description of weakly perturbed stable limit cycle oscillations defined by Eqs. (3a), (3b), and (4) can essentially be simplified by using the phase-reduction method. The application of this method to the systems with time delays is presented in our recent paper [25]. The consideration in Ref. [25] was restricted to the systems with a single time delay. In Appendix A, we present its straightforward extension to the case of multiple time delays. According to the results presented in Appendix A, the dynamics of the weakly perturbed EDFC systems (3a), (3b), and (4) can be reduced to the phase dynamics as follows:

$$\dot{\varphi}(t) = 1 + \varepsilon\mathbf{z}^T[\varphi(t)]\boldsymbol{\psi}(t) + O(\varepsilon^2). \tag{6}$$

Here, $\varphi(t)$ is a scalar variable that defines the phase of oscillations, and $\mathbf{z}$ is an infinitesimal PRC of the system. The PRC $\mathbf{z} = (z_1,\ldots,z_n)^T$ represents an n-dimensional T-periodic vector function $\mathbf{z}(\varphi) = \mathbf{z}(\varphi+T)$ that satisfies the adjoint equation,

$$\dot{\mathbf{z}}^T(t) = -\mathbf{z}^T(t)\mathbf{A}_0(t) + \mathbf{z}^T(t)\mathbf{W}(t)\mathbf{K}\mathbf{V}(t) - \sum_{j=1}^{M}\mathbf{z}^T(t+jT)\mathbf{W}(t)\mathbf{K}\mathbf{P}\mathbf{R}^{j-1}\mathbf{V}(t), \tag{7}$$

where

$$\mathbf{A}_0(t) = D_1\mathbf{f}[\boldsymbol{\xi}(t),0], \tag{8a}$$
$$\mathbf{W}(t) = D_2\mathbf{f}[\boldsymbol{\xi}(t),0], \tag{8b}$$
$$\mathbf{V}(t) = D\mathbf{g}[\boldsymbol{\xi}(t)] \tag{8c}$$

are the T-periodic matrices. The matrix $\mathbf{A}_0(t)$ is the Jacobian matrix of the control-free system estimated on the UPO $\boldsymbol{\xi}(t)$, where D_1 is the vector derivative of the function $\mathbf{f}(\mathbf{x},\mathbf{u})$ with respect to the first argument. The symbol D_2 in the definition of the matrix $\mathbf{W}(t)$ denotes the vector derivative of the function $\mathbf{f}(\mathbf{x},\mathbf{u})$ with respect to the second argument. The matrix $\mathbf{V}(t)$ represents the vector derivative of the function $\mathbf{g}(\mathbf{x})$ that relates the output variable $\mathbf{s}$ with the state variable $\mathbf{x}$. Equation (7) is derived from Eqs. (A5) and (A3) taking into account that the multiple delay times in our case are as follows: $\tau_j = jT$, $j = 1,\ldots,M$.

To find the PRC of the system, we have to solve Eq. (7) with the requirement of the periodicity $\mathbf{z}(t+T) = \mathbf{z}(t)$. It is easy to see that, for any T-periodic $\mathbf{z}(t)$ function, the last two terms in Eq. (7) vanish since $\sum_{j=1}^{M} \mathbf{z}^T(t+jT)\mathbf{W}(t)\mathbf{KPR}^{j-1}\mathbf{V}(t) = \mathbf{z}^T(t)\mathbf{W}(t)\mathbf{KV}(t)$. Therefore, the PRC of the controlled system also satisfies the adjoint equation of the control-free system,

$$\dot{\mathbf{z}}^T(t) = -\mathbf{z}^T(t)\mathbf{A}_0(t). \tag{9}$$

This equation is independent of the control $\mathbf{K}$ and memory $\mathbf{R}$ matrices. It means that the profile of the PRC of the controlled system is invariant with respect to the variation in $\mathbf{K}$ and $\mathbf{R}$. However, the amplitude of the PRC does depend on these matrices. Let us say we have two different PRCs $\mathbf{z}^{(1)}(t)$ and $\mathbf{z}^{(2)}(t)$, corresponding to two different selections of the control and memory matrices $(\mathbf{K}^{(1)},\mathbf{R}^{(1)})$ and $(\mathbf{K}^{(2)},\mathbf{R}^{(2)})$, then from the invariancy of the profile, it follows that these PRCs are proportional to each other: $\mathbf{z}^{(2)}(t) \propto \mathbf{z}^{(1)}(t)$.

In the following, we denote the PRC of a UPO of the control-free system as $\boldsymbol{\rho}(t)$ and treat it as a basic PRC. Then, we can express the PRC $\mathbf{z}(t)$ of the controlled system for any choice of $(\mathbf{K},\mathbf{R})$ through this basic PRC,

$$\mathbf{z}(t) = \alpha(\mathbf{K},\mathbf{R})\boldsymbol{\rho}(t). \tag{10}$$

The proportionality coefficient $\alpha(\mathbf{K},\mathbf{R})$ can be obtained from the initial condition (A6), which for our system, takes the form

$$\mathbf{z}^T(0)\dot{\boldsymbol{\xi}}(0) + \sum_{j=1}^{M} j\int_{-T}^{0} \mathbf{z}^T(\vartheta)\mathbf{B}_j(\vartheta)\dot{\boldsymbol{\xi}}(\vartheta)d\vartheta = 1, \tag{11}$$

where

$$\mathbf{B}_j(\vartheta) = \mathbf{W}(\vartheta)\mathbf{KPR}^{j-1}\mathbf{V}(\vartheta). \tag{12}$$

The PRC of the UPO of the control-free system is a periodic function $\boldsymbol{\rho}(t) = \boldsymbol{\rho}(t+T)$ that satisfies the following equation and initial condition:

$$\dot{\boldsymbol{\rho}}^T(t) = -\boldsymbol{\rho}^T(t)\mathbf{A}_0(t), \tag{13a}$$
$$\boldsymbol{\rho}^T(0)\dot{\boldsymbol{\xi}}(0) = 1. \tag{13b}$$

The sum in Eq. (11) can be determined analytically,

$$\sum_{j=1}^{M} j\mathbf{R}^{j-1} = [\mathbf{I} + M\mathbf{R}^{M+1} - (M+1)\mathbf{R}^M](\mathbf{I}-\mathbf{R})^{-2}, \tag{14}$$

and then, Eq. (11) simplifies to

$$\mathbf{z}^T(0)\dot{\boldsymbol{\xi}}(0) + \int_{-T}^{0} \mathbf{z}^T(\vartheta)\mathbf{W}(\vartheta)\mathbf{KQV}(\vartheta)\dot{\boldsymbol{\xi}}(\vartheta)d\vartheta = 1, \tag{15}$$

where

$$\mathbf{Q} = (\mathbf{I}-\mathbf{R})^{-1} - M\mathbf{R}^M(\mathbf{I}-\mathbf{R}^M)^{-1}. \tag{16}$$

To derive the expression for the above coefficient of proportionality $\alpha(\mathbf{K},\mathbf{R})$, we substitute Eq. (10) into Eq. (15) and use the condition (13b). Then, we take the limit $M \to \infty$. For $M \to \infty$, the matrix (16) becomes $\mathbf{Q} = (\mathbf{I}-\mathbf{R})^{-1}$, and we finally obtain

$$\alpha(\mathbf{K},\mathbf{R}) = \left[1 + \sum_{r=1}^{k}\sum_{p=1}^{l} \frac{K_{rp}C_{pr}}{1-R_p}\right]^{-1}. \tag{17}$$

Here, we introduced an $l \times k$ matrix $\mathbf{C}$, whose elements are defined by a double sum of the following integral:

$$C_{pr} = \sum_{i=1}^{n}\sum_{s=1}^{n} \int_{-T}^{0} \rho_i(\vartheta)W_{ir}(\vartheta)V_{ps}(\vartheta)\dot{\xi}_s(\vartheta)d\vartheta. \tag{18}$$

The matrix $\mathbf{C}$ captures all inherent properties of the controlled system that define the variation in the PRC amplitude in response to the variation in the matrices $(\mathbf{K},\mathbf{R})$.

Equations (10) and (17) provide a convenient way to obtain the PRC of the controlled system for any set of matrices $(\mathbf{K},\mathbf{R})$ using the knowledge of the basic PRC $\boldsymbol{\rho}(t)$ defined by Eqs. (13). Unfortunately, Eq. (13a) is difficult to employ for a numerical computation of the basic PRC $\boldsymbol{\rho}(t)$ since this PRC corresponds to the UPO of the control-free system. Any UPO of a chaotic system has Lyapunov exponents with positive and negative real parts so that Eq. (13a) is unstable for both the backwards and forwards integration. To avoid this problem, we do not use Eq. (13a) for the numerical computation of $\boldsymbol{\rho}(t)$. Instead, we employ Eq. (7), which is stable for the backwards integration, provided the matrices $(\mathbf{K},\mathbf{R})$ are chosen from the domain of stability of the given UPO (cf. Ref. [25]). To compute the basic PRC $\boldsymbol{\rho}(t)$, we proceed as follows. We choose some fixed values of the matrices $(\mathbf{K}^{(1)},\mathbf{R}^{(1)})$ at which, the controlled orbit is stable and, using arbitrary initial conditions, integrate Eq. (7) backwards in time until it converges to a periodic solution. Then, the basic PRC $\boldsymbol{\rho}(t)$ is obtained by normalizing the amplitude of this solution according to Eq. (13b).

The knowledge of the PRC $\mathbf{z}(t)$ of the controlled orbit allows us to easily analyze the phase dynamics of the complex delay differential equation (DDE) system (3) in the presence of any weak time-dependent perturbations using a simple scalar ordinary differential Eq. (6). In the next section, we utilize these results in order to derive an analytical expression for the period Θ of the stabilized orbit in the presence of a small mismatch when the delay time τ differs slightly from the period T of the target UPO.

IV. EDFC SYSTEM FOR $\tau \neq T$: DERIVATION OF AN EXPRESSION FOR THE PERIOD Θ

In this section, we consider Eqs. (2) without external perturbation but suppose that $\tau \neq T$. Our aim is to derive an analytical expression for the period Θ of the stabilized orbit in the case of a small time delay mismatch. The key idea of our approach is based on splitting the control variable into nonmismatched and mismatched components. The nonmismatched component stabilizes the UPO, whereas, the mismatched component induces a small perturbation that can be treated by the above presented phase-reduction theory.

Similar to the previous section, we consider the EDFC with a finite number of delay terms by replacing Eq. (2c) with

$$\tilde{\mathbf{u}}(t) = \mathbf{K}\left[\mathbf{P}\sum_{j=1}^{M}\mathbf{R}^{j-1}\mathbf{s}(t-j\tau) - \mathbf{s}(t)\right]. \tag{19}$$

We represent the delay time in the form $\tau = T + \varepsilon$, where

$$\varepsilon = \tau - T \tag{20}$$

is a small mismatch. Substituting this expression for τ in Eq. (19) and expanding it with respect to ε, we get

$$\tilde{\mathbf{u}}(t) = \tilde{\mathbf{u}}_0(t) - \varepsilon\mathbf{KP}\sum_{j=1}^{M} j\mathbf{R}^{j-1}D\mathbf{g}[\mathbf{x}(t-jT)]\dot{\mathbf{x}}(t-jT) + O(\varepsilon^2). \tag{21}$$

Here, $\tilde{\mathbf{u}}_0(t)$ is a nonmismatched part of the control variable defined by Eq. (4). The remaining terms in Eq. (21) represent the mismatched part of the control variable. Now, substituting Eq. (21) into Eq. (2a) and expanding it with respect to ε up to the first-order terms, we reveal that the systems (2a), (2b), and (19) transform exactly to the forms (3a), (3b), and (4) with

$$\boldsymbol{\psi}(t) = -D_2\mathbf{f}[\mathbf{x}(t),\tilde{\mathbf{u}}(t)]\mathbf{KP}\sum_{j=1}^{M} j\mathbf{R}^{j-1} \times D\mathbf{g}[\mathbf{x}(t-jT)]\dot{\mathbf{x}}(t-jT). \tag{22}$$

When considering the phase dynamics of the stabilized orbit $\boldsymbol{\xi}(t)$, in Eq. (22), we can substitute $\mathbf{x}(t) = \boldsymbol{\xi}(t)$ and can treat $\boldsymbol{\psi}(t)$ as an external perturbation (cf. Ref. [26]). Then, taking the periodicity of $\boldsymbol{\xi}(t)$ into account, Eq. (22) simplifies to

$$\boldsymbol{\psi}(t) = -\mathbf{W}(t)\mathbf{KQV}(t)\dot{\boldsymbol{\xi}}(t). \tag{23}$$

Taking the limit $M \to \infty$, Eq. (23) transforms to

$$\boldsymbol{\psi}(t) = -\mathbf{W}(t)\mathbf{K}(\mathbf{I}-\mathbf{R})^{-1}\mathbf{V}(t)\dot{\boldsymbol{\xi}}(t). \tag{24}$$

Now, we can utilize all the results of the previous section, which have been derived for an arbitrary external perturbation $\boldsymbol{\psi}(t)$. In our case, the external perturbation has the particular form (24), and the parameter ε is defined by Eq. (20).

The solution of the phase Eq. (6) has the form $\varphi = t + O(\varepsilon)$, and therefore, it can be alternatively written as

$$\dot{\varphi}(t) = 1 + \varepsilon\mathbf{z}^T(\varphi)\boldsymbol{\psi}(\varphi) + O(\varepsilon^2). \tag{25}$$

The period Θ of the target orbit in the presence of a small time delay mismatch can be estimated as follows:

$$\Theta = \int_0^T \frac{d\varphi}{1+\varepsilon\mathbf{z}^T(\varphi)\boldsymbol{\psi}(\varphi)} + O(\varepsilon^2) = T - \varepsilon\int_0^T \mathbf{z}^T(\varphi)\boldsymbol{\psi}(\varphi)d\varphi + O(\varepsilon^2). \tag{26}$$

The integral in this equation can be expressed through the coefficient $\alpha(\mathbf{K},\mathbf{R})$ introduced in the previous section,

$$\int_0^T \mathbf{z}^T(\varphi)\boldsymbol{\psi}(\varphi)d\varphi = \alpha(\mathbf{K},\mathbf{R}) - 1. \tag{27}$$

This result follows from Eqs. (10), (17), (18), and (24). Substituting Eq. (27) into Eq. (26) and using Eqs. (17) and (20), we finally obtain the following analytical expression for the period $\Theta = \Theta(\mathbf{K},\mathbf{R},\tau)$:

$$\Theta = T + (\tau - T)\frac{\sum_{r=1}^{k}\sum_{p=1}^{l} K_{rp}C_{pr}/(1-R_p)}{1+\sum_{r=1}^{k}\sum_{p=1}^{l} K_{rp}C_{pr}/(1-R_p)} + O[(\tau-T)^2]. \tag{28}$$

Equation (1), derived in Ref. [24], is the special case of Eq. (28). If we take the zero memory matrix $\mathbf{R} = 0$ and assume that the control matrix has only one nonzero element $K_{11} = K$, then Eq. (28) transforms to Eq. (1) with $\kappa = -1/C_{11}$.

V. NUMERICAL DEMONSTRATIONS

Now, we support the validity of the above theoretical results and demonstrate their efficiency using numerical simulations. For numerical analysis, we choose the Rössler system [27] with the standard parameter values,

$$\dot{x}_1(t) = -x_2(t) - x_3(t) + u_1(t), \tag{29a}$$

$$\dot{x}_2(t) = x_1(t) + 0.2x_2(t) + u_2(t), \tag{29b}$$

$$\dot{x}_3(t) = 0.2 + x_3(t)[x_1(t) - 5.7], \tag{29c}$$

$$u_p(t) = K_{pp}\left[(1-R_p)\sum_{j=1}^{\infty}R_p^{j-1}x_p(t-j\tau) - x_p(t)\right] \quad \text{for } p = 1,2. \tag{29d}$$

Here, we apply the control perturbation to the first two equations of the Rössler system using different values of the memory parameter and different values of the coupling strength. In this case, the output variable is a two-dimensional vector $\mathbf{s}(t) = [x_1(t),x_2(t)]^T$, and both matrices $\mathbf{K}$ and $\mathbf{R}$ are diagonal: $\mathbf{K} = \mathrm{diag}(K_{11},K_{22})$ and $\mathbf{R} = \mathrm{diag}(R_1,R_2)$. We rewrite Eq. (28) in the form

$$\Theta = T + (\tau - T)\Delta, \tag{30}$$

where the parameter Δ for this particular case is as follows:

$$\Delta = \frac{K_{11}C_{11}/(1-R_1) + K_{22}C_{22}/(1-R_2)}{1 + K_{11}C_{11}/(1-R_1) + K_{22}C_{22}/(1-R_2)}. \tag{31}$$

In the following, we like to check whether Eq. (31) correctly describes the dependence of Δ on the control parameters K_{11}, K_{22}, R_1, and R_2. When analyzing the dependence of Δ on the coupling strengths K_{11} and K_{22}, it is convenient to introduce a ratio $\Delta/(1-\Delta)$ since, according to Eq. (31), it depends linearly on these parameters,

$$\frac{\Delta}{1-\Delta} = \frac{K_{11}C_{11}}{1-R_1} + \frac{K_{22}C_{22}}{1-R_2}. \tag{32}$$

As a first demonstration, we consider the stabilization of a period-two UPO of the Rössler system with the period $T \approx 11.758\,63$. We compute the PRC $\boldsymbol{\rho}(t)$ of the control-free UPO as described in Sec. III and estimate the parameters

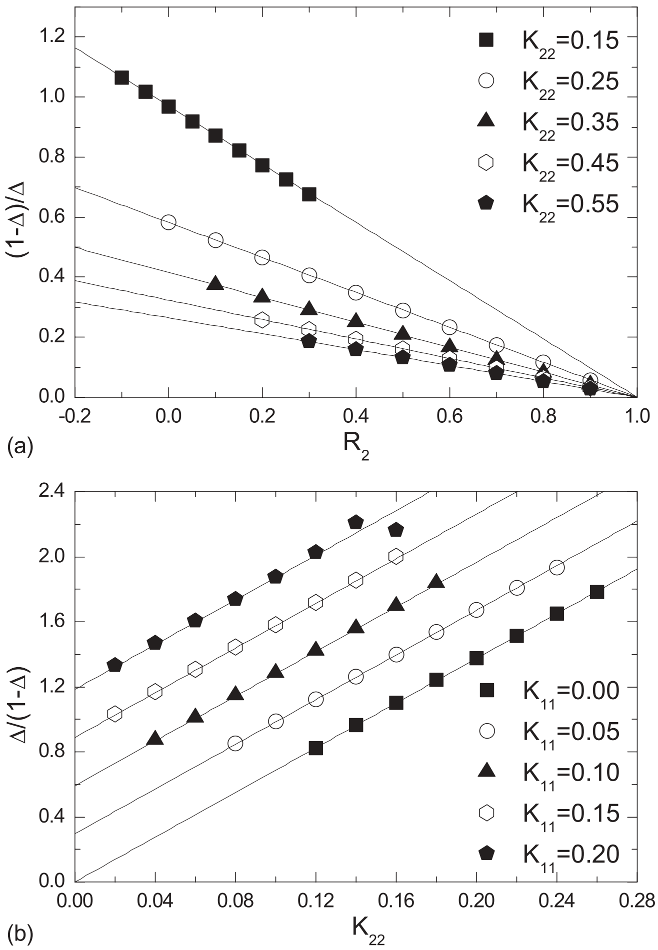

FIG. 1. Numerical results for the period-two UPO of the Rössler system. (a) The parameter $(1-\Delta)/\Delta$ as a function of a memory parameter R_2 for a fixed $K_{11}=0$ and different values of the parameter K_{22}. (b) The parameter $\Delta/(1-\Delta)$ as a function of control amplitude K_{22} for the fixed $R_1=R_2=0$ and different values of K_{11}. The symbols represent the results of the direct numerical simulation of the original system (29), whereas, the solid lines show the analytical results obtained from Eq. (32).

$C_{11}=5.9119$ and $C_{22}=6.8694$ according to Eq. (18). Then, we compute the ratio (32) and, in Fig. 1(a), plot (the solid lines) its inverse $(1-\Delta)/\Delta$ as a function of the memory parameter R_2 for a fixed $K_{11}=0$ and different values of the parameter K_{22}. In part (b) of Fig. 1, we plot $\Delta/(1-\Delta)$ as a function of control amplitude K_{22} for the fixed $R_1=R_2=0$ and different values of K_{11}. To illustrate the validity of these theoretical results, the parameter Δ has alternatively been estimated by the direct numerical simulation of the original system (29). To this end, the delay time τ has been chosen close to the period T of the UPO so that $\tau-T=0.02$ and the period Θ of the stabilized orbit has been estimated numerically. Then, the parameter Δ has been computed according to Eq. (30) as $\Delta=(\Theta-T)/(\tau-T)$. These results are shown in the figure by symbols. Note that the latter algorithm is applicable only in the domain of control parameters where the stabilization of the target UPO is successful. We see that the analytical expression (31) predicts the dependence of Δ on the control parameters well.

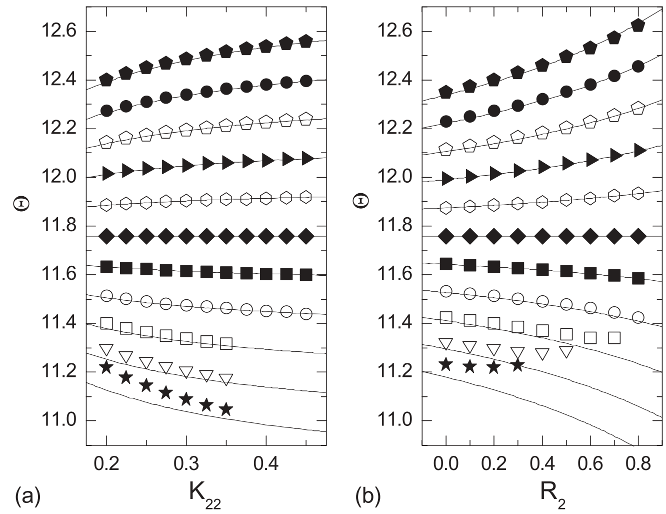

FIG. 2. The period Θ of the control signal as a function of control parameters (a) K_{22} and (b) R_2 for various delay times; from bottom to top, $\tau-T=-1.0,\ -0.8,\ -0.6,\ -0.4,\ -0.2,$ 0, 0.2, 0.4, 0.6, 0.8, and 1.0. Other control parameters are as follows: (a) $K_{11}=0$, $R_2=0.2$ and (b) $K_{11}=0$, $K_{22}=0.2$. The symbols represent the results of the direct numerical simulation of the original system (29), whereas, the solid curves show the analytical results obtained from Eqs. (30) and (31). Some data points are missed because the fixed point has been stabilized instead of the periodic orbit.

To estimate the size of the mismatch $\tau-T$ for which Eqs. (30) and (31) are valid, in Fig. 2, we plot the dependence of Θ on K_{22} [part (a)] and R_2 [part (b)] for different values of the parameter $\tau-T$. Again, the solid curves show the analytical results obtained from Eqs. (30) and (31), whereas, the symbols correspond to the direct numerical simulation of Eqs. (29). As seen from the figure, the analytical results are valid for the values of the relative mismatch $(\tau-T)/T$ up to 10%.

In Fig. 3, we demonstrate the verification of our theoretical results for a high-period UPO for which, the usual DFC algorithm ($\mathbf{R}=0$) does not work. We consider the stabilization of a period-eight UPO with the period $T\approx 47.026\,164\,8$. The profile of the x_2 component of the UPO and the ρ_2 component of its PRC are shown in parts (a) and (b), respectively. The dependence of the ratio (32) on the coupling strength K_{22} for $K_{11}=0$ and different values of the memory parameter R_2 is shown in part (c). The parameter $C_{22}=27.4483$ has been estimated from Eq. (18) using the computed PRC $\rho_2(\varphi)$. Again, we observe a good coincidence of the analytical results given by Eq. (32) with the results of the direct numerical simulation of the original system (29).

VI. CONCLUSIONS

We have considered a general class of multiple-input multiple-output systems subjected to an EDFC force in the case when the delay time differs slightly from the period of the UPO of the control-free system. We have derived an analytical expression, which shows, in an explicit form, how the period of stabilized orbit changes when varying the delay time and the parameters of the control and memory matrices. This result is important for the experimental implementation of

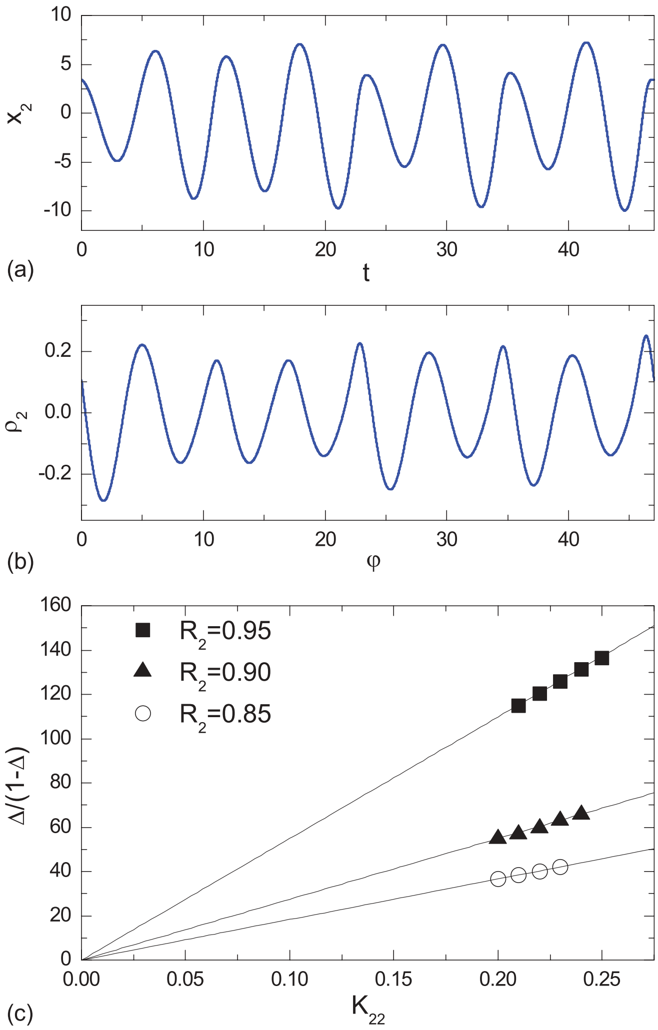

FIG. 3. (Color online) Numerical results for the period-eight UPO of the Rössler system. (a) The profile of the $x_2(t)$ component of the UPO. (b) The second PRC component $\rho_2(\varphi)$ of the UPO. (c) The dependence of the ratio $\Delta/(1-\Delta)$ on the control amplitude K_{22} for $K_{11}=0$ and different values of the memory parameter R_2. The symbols represent the results of the direct numerical simulation of the original system (29), whereas, the solid lines show the analytical results obtained from Eq. (32).

the EDFC algorithm since it can facilitate the determination of an unknown period of control-free UPOs. Using an analytical relationship between the period of the control signal and the control parameters, the unknown period of the control-free UPO can be determined from only a few experimental measurements (cf. Ref. [24]).

Our approach is based on the phase-reduction theory adapted to systems with time delays [25]. We have reduced the original EDFC system described by delay differential equations to a simple scalar equation, which defines the dynamics of the phase. We have also derived an equation for the PRC of controlled orbit and have shown that its profile is independent of the control and memory matrices. This fact allows us to express the PRCs of the controlled UPO through the PRC of the control-free UPO (the basic PRC). Although the basic PRC corresponds to the unstable orbit, we have formulated an algorithm for its numerical computation.

The approach based on the PRC theory is very universal. Here, we have demonstrated its benefit to obtain the period of the control signal in the presence of a small time delay mismatch. The knowledge of the basic PRC and its relationship with the PRCs of the controlled system with arbitrary control parameters allows a simple investigation of the influence of any weak time-dependent perturbations on the controlled system. Furthermore, our results are of particular interest from the plain theoretical point of view as well since dynamics with time delay plays an important role in various fields of science.

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APPENDIX A: PHASE-REDUCTION THEORY FOR SYSTEMS WITH MULTIPLE TIME DELAYS

The classical phase-reduction theory is usually formulated for a weakly perturbed limit cycle oscillator described by the ordinary differential equations (ODEs). Recently, we have extended this approach to the systems described by DDEs with a single time delay [25]. Two methods of derivation of the phase-reduced equation have been presented. The first (heuristic) method is based on physical arguments and uses the representation of the delay term by a delay line, which we model by an advection equation. By discretizing the space variable of the advection equation, we come to a finite set of ODEs and apply the classical PRC theory. Then, we return to a continuous limit and obtain the PRC for the original DDE. The second (direct) method deals directly with the DDEs without recourse to the ODEs; it is based on the expansion of the solution close to the limit cycle in the Floquet modes of the linearized system. Both methods lead to the same result, but the second method seems uncertain for some systems. If the linearized system of the DDEs possesses small solutions, the completeness of the expansion in the Floquet modes is not guaranteed [28]. We discuss this issue in more detail in Appendix B.

In this paper, we need an extension of the phase-reduction theory to the systems with multiple time delays. The derivation of the phase-reduced equation for the DDEs with multiple time delays is rather long but is essentially the same as in the case of a single delay described in Ref. [25]. Here, we do not present the details of this derivation but outline only the main results.

Consider a weakly perturbed limit cycle oscillator described by DDEs with multiple constant delays,

$$\dot{\mathbf{x}}(t) = \mathbf{F}[\mathbf{x}(t),\mathbf{x}(t-\tau_1),\ldots,\mathbf{x}(t-\tau_M)] + \varepsilon\boldsymbol{\psi}(t). \quad \text{(A1)}$$

Here, $\mathbf{x}=(x_1,\ldots,x_n)^T$ is an n-dimensional vector, $\tau_1,\ldots\tau_M$ are the constant time delays, and $\varepsilon\boldsymbol{\psi}(t)=\varepsilon[\psi_1(t),\ldots,\psi_n(t)]^T$ represents a small time-dependent perturbation, where ε is a small parameter $|\varepsilon|\ll 1$. We assume that, for $\varepsilon=0$, the system has a stable limit cycle solution $\mathbf{x}=\boldsymbol{\xi}(t)$ with a period T: $\boldsymbol{\xi}(t)=\boldsymbol{\xi}(t+T)$. In the absence of external perturbation ($\varepsilon=0$), the dynamics of the system close to the limit cycle is described by the linearized equation,

$$\delta\dot{\mathbf{x}}(t) = \mathbf{A}(t)\delta\mathbf{x} + \sum_{j=1}^{M} \mathbf{B}_j(t-\tau_j)\delta\mathbf{x}(t-\tau_j), \quad \text{(A2)}$$

where $\delta\mathbf{x}(t) = \mathbf{x}(t) - \boldsymbol{\xi}(t)$. The matrices $\mathbf{A}(t)$ and $\mathbf{B}_j(t)$, $j = 1,\ldots,M$ are the T-periodic Jacobian matrices defined as the vector derivatives of the function $\mathbf{F}[\mathbf{x}(t),\mathbf{x}(t-\tau_1),\ldots,\mathbf{x}(t-\tau_M)]$ with respect to the first (D_1) and $(j+1)$-th (D_{j+1}) argument, estimated on the limit cycle of the unperturbed system,

$$\mathbf{A}(t) = D_1\mathbf{F}[\boldsymbol{\xi}(t),\boldsymbol{\xi}(t-\tau_1),\ldots,\boldsymbol{\xi}(t-\tau_M)], \quad \text{(A3a)}$$

$$\mathbf{B}_j(t) = D_{j+1}\mathbf{F}[\boldsymbol{\xi}(t),\boldsymbol{\xi}(t-\tau_1),\ldots,\boldsymbol{\xi}(t-\tau_M)], \quad j = 1,\ldots,M. \quad \text{(A3b)}$$

The phase reduction in Eq. (A1) can be performed by expanding its solution in Floquet modes of the linearized system (A2) (cf. Ref. [25] for details). As a result, the DDE system (A1) is reduced to a single scalar equation that defines the dynamics of the phase,

$$\dot{\varphi}(t) = 1 + \varepsilon\mathbf{z}^T[\varphi(t)]\boldsymbol{\psi}(t) + O(\varepsilon^2). \quad \text{(A4)}$$

Here, $\mathbf{z} = (z_1,\ldots,z_n)^T$ is an n-dimensional T-periodic vector function $\mathbf{z}(\varphi+T) = \mathbf{z}(\varphi)$, referred to as an infinitesimal PRC. The PRC is a periodic solution of the adjoint equation,

$$\dot{\mathbf{z}}^T(t) = -\mathbf{z}^T(t)\mathbf{A}(t) - \sum_{j=1}^{M} \mathbf{z}^T(t+\tau_j)\mathbf{B}_j(t+\tau_j), \quad \text{(A5)}$$

with the initial condition,

$$\mathbf{z}^T(0)\dot{\boldsymbol{\xi}}(0) + \sum_{j=1}^{M}\int_{-\tau_j}^{0} \mathbf{z}^T(\tau_j+\vartheta)\mathbf{B}_j(\tau_j+\vartheta)\dot{\boldsymbol{\xi}}(\vartheta)d\vartheta = 1. \quad \text{(A6)}$$

The adjoint equation (A5) represents a difference-differential equation of the advanced type. Although this equation is unstable, its periodic solution can be obtained numerically by a backward integration [25] in a similar way as performed for the case of ordinary differential equations [29]. Since the matrices $\mathbf{A}(t)$ and $\mathbf{B}_j(t)$ are usually unavailable in an analytical form, their values are estimated from a forward integration of the unperturbed system (A1).

APPENDIX B: ON AN INFLUENCE OF SMALL SOLUTIONS

The direct method of derivation of the phase-reduced Eq. (A4) for the system (A1) is based on the assumption that any solution close to the limit cycle can be expanded in Floquet modes of the linearized system (A2). The system (A2) represents a nonautonomous DDE whose time delays $\tau_j = jT$, $j = 1,\ldots,M$ are multiples of the period of the matrices $\mathbf{A}(t) = \mathbf{A}(t+T)$ and $\mathbf{B}_j(t) = \mathbf{B}_j(t+T)$. As shown in Refs. [30,31], such a system may admit small solutions, which decay faster than any exponent, i.e., $\lim_{t\to\infty}\delta\mathbf{x}(t)e^{kt} = 0$ for all $k \in \mathbb{R}$. If the system possesses small solutions, the completeness of expansion in the Floquet modes is not guaranteed. In Ref. [28], it is shown that only the properties of the matrix $\mathbf{B}_M(t)$ are relevant to the existence of small solutions. Let us denote the eigenvalues of the matrix $\mathbf{B}_M(t)$ as $\lambda_i(t)$, $i = 1,\ldots,n$. The small solutions appear if some of the eigenvalues $\lambda_i(t)$ at a moment t_0 cross the origin in a complex plane [32],

$$\mathrm{Re}\,\lambda_i(t_0+)\mathrm{Re}\,\lambda_i(t_0-) < 0, \qquad \lambda_i(t_0) = 0. \quad \text{(B1)}$$

We stress that the examples presented in Sec. V do not contain small solutions since the matrices $\mathbf{B}_j(t)$ are independent of time and do not meet the criterion (B1). Note that the small solutions will not appear for the most popular DFC schemes when the control perturbation is applied as an external force with a constant control matrix.

To check whether our analytical results remain valid for the case when the system possesses small solutions, we consider the following specific example based on the Landau-Stuart oscillator:

$$\dot{x}_1(t) = x_1(t)\left[1 - x_1^2(t) - x_2^2(t)\right] - x_2(t)\left[u(t) + x_1^2(t) + x_2^2(t)\right], \quad \text{(B2a)}$$

$$\dot{x}_2(t) = x_2(t)\left[1 - x_1^2(t) - x_2^2(t)\right] + x_1(t)\left[x_1^2(t) + x_2^2(t)\right], \quad \text{(B2b)}$$

$$s(t) = g[x_1(t),x_2(t)] = x_1^2(t), \quad \text{(B2c)}$$

$$u(t) = K\,[s(t-\tau) - s(t)]\,. \quad \text{(B2d)}$$

Here, $u(t)$, $s(t)$, and K are the scalars and the memory parameter $R = 0$. Unlike the examples presented in Sec. V, here, the control-free system ($K = 0$) has a stable limit cycle $\xi_1(t) = \cos(t)$ and $\xi_2(t) = \sin(t)$, but the control term may induce small solutions. For our system, Eqs. (8b) and (8c) are as follows:

$$\mathbf{W}(t) = [-\xi_2(t) \quad 0]^T, \quad \text{(B3a)}$$

$$\mathbf{V}(t) = [2\xi_1(t) \quad 0]. \quad \text{(B3b)}$$

According to Eq. (12), the matrix $\mathbf{B}_1(t)$, responsible for the existence of small solutions, has the form

$$\mathbf{B}_1(t) = \begin{pmatrix} -2K\xi_1(t)\xi_2(t) & 0 \\ 0 & 0 \end{pmatrix}. \quad \text{(B4)}$$

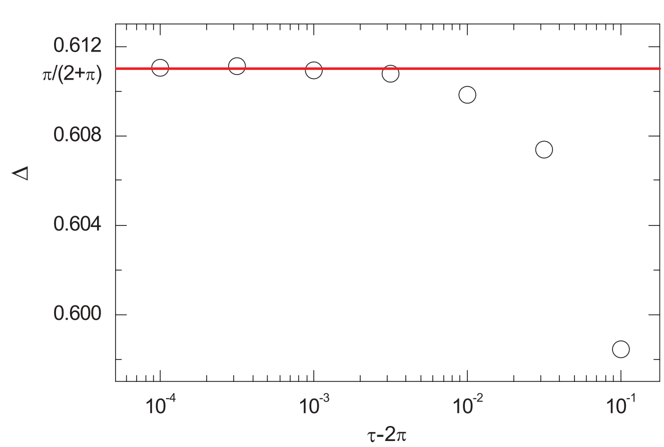

FIG. 4. (Color online) Comparison of numerical simulations of the system (B2) with the analytical result (B5). The cycles represent numerically computed values of Δ for different mismatches, whereas, the straight line shows the analytical result.

One of the eigenvalues of the matrix $\mathbf{B}_1(t)$ crosses the origin of the complex plane four times per period. Therefore, according to the criterion (B1), our system has small solutions. The presence of small solutions in the system has also been confirmed numerically by using the algorithm described in Refs. [33,34].

A nice property of the system (B2) is that the basic PRC (13) can be obtained analytically: $\rho_1(\vartheta) = -\sqrt{2}\sin(\vartheta - \pi/4)$ and $\rho_2(\vartheta) = \sqrt{2}\cos(\vartheta - \pi/4)$. The parameter Δ in Eq. (30) can be defined analytically as well. For the control gain $K = 1$, the value of this parameter is as follows:

$$\Delta = \pi/(2+\pi). \tag{B5}$$

In Fig. 4, we show that numerical simulations of the system (B2) converge to the analytical result (B5) for small time delay mismatches. This example demonstrates that our analytical results remain valid even for the system possessing small solutions. Additional investigations are needed to ascertain whether this conclusion is valid in general.

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