PHASE LEAD SYNCHRONIZATION OF CHAOTIC OSCILLATORS

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Abstract
We present the new effect of phase lead synchronization in unidirectionally coupled (”master–slave” or drive–response configuration) chaotic oscillators. Here the phases of the coupled systems are locked in such a way that the response phase is ahead of the drive phase. This phenomenon appearers when the response system is faster than the drive system. The effect can be used for forecasting the chaotic dynamics of the drive system. We demonstrate this phenomenon for unidirectionally coupled nonidentical chaotic Rössler systems. The results of both analytical and numerical investigations are presented.

Key words
Phase synchronization, anticipating synchronization, chaotic oscillations.

1 Introduction

Though long ago the notion of chaos was associated with the absolutely unpredictable state or process [Lorenz, 1963] the things have been changed due to the progress in computer technology and, in 2000, a way of forecasting of chaotic dynamics was introduced by Voss [Voss, 2000]. The proposed method relies on the phenomenon of anticipating synchronization.

The phenomenon of synchronization refers to the collective timing of coupled systems and manifests itself in physical, chemical as well as biological systems [Pikovsky, Rosemblum and Kurths, 2001]. The notion of synchronization was generalized and applied to different chaotic systems [Afraimovich, Verichev, and Rabinovich, 1983; Boccaletti et al., 2002; Pecora and Carroll, 1990]. Anticipating synchronization refers to a particular regime, which appears in unidirectionally coupled systems in a drive–response configuration. In this regime, two dynamical systems synchronize in such a way that the response system anticipates the trajectory of the drive. Anticipating synchronization has been studied theoretically and experimentally in many systems, e. g., in chaotic semiconductor lasers in electronic circuits, in excitable systems, in coupled inertial ratchets, and in neural networks [Ciszak et al., 2003; Ciszak et al., 2004; Kostur et al., 2005; Masoller, 2001; Voss, 2002].

Two principal schemes have been proposed in order to achieve anticipated synchronization. The first scheme implies the presence of delay in the master system [Voss, 2000]. The second, refereed to as the delay coupling scheme [Voss, 2001], deserves more attention. This is because of the fact that the master system does not need to have delay term. The delay term is incorporated into the slave system so that the anticipating time can be varied without influence the dynamics of the master system. However, that scheme requires some constraints on the anticipation time and coupling strength for the synchronization solution to be stable [Voss, 2001; Calvo et al., 2004]. Recently we have shown that the prediction time of such a scheme can bee essentially enlarged via an appropriate construction of the coupling matrix [Pyragas and Pyragienė, 2008; Pyragas and Pyragienė, 2010].

In the present work, we have detected a more simple and more universal way for the prediction of chaotic dynamics. We show that anticipating synchronization may appear in a very simple coupling scheme without delay terms in either drive or response systems. The effect is based on a phase synchronization (PS) first described in [Rosenblum, Pikovsky and Kurths, 1996; Rosenblum, Pikovsky and Kurths, 1997]. In these papers, the authors have considered the case of mutual coupling between two chaotic oscillators and demonstrated that their phases may lock at a sufficiently large coupling strength. Contrary to this, we consider the case of unidirectional coupling and show that the phase synchronization may appear in such a way that the phase of the response system outruns the phase of the drive system. As a result, the response systems anticipates the dynamics of the drive system.

In the following, we first introduce the notion of the phase lead synchronization for a simple system of unidirectionally coupled phase oscillators. Then we demonstrate that a similar phase lead synchronization effect takes place in unidirectionally coupled chaotic Rössler systems when the response system is faster than the drive. We perform both the analytical and numerical analysis of this phenomenon and show that it can be used to forecast the chaotic dynamics of the drive system.

2 Phase lead synchronization in a simple model

Consider a simple model of unidirectionally coupled phase oscillators:

$$\dot{\varphi}_1 = \omega_1, \tag{1a}$$
$$\dot{\varphi}_2 = \omega_2 + \varepsilon \sin(\varphi_1 - \varphi_2), \tag{1b}$$

where φ_1 and φ_2 are the phases of the oscillators, ω_1 and ω_2 are their natural frequencies and $\varepsilon > 0$ is the coupling strength. We suppose that the response oscillator is faster then the drive, $\omega_2 > \omega_1$. The phase difference $\Delta = \varphi_2 - \varphi_1$ between the oscillators is governed by the Adler equation:

$$\dot{\Delta} = (\omega_2 - \omega_1) - \varepsilon \sin\Delta. \tag{2}$$

This system has a stable fixed point

$$\Delta_0 = \arcsin\left(\frac{\omega_2 - \omega_1}{\varepsilon}\right) > 0 \tag{3}$$

of positive sign provided $\varepsilon > \omega_2 - \omega_1$. Thus if this condition is fulfilled we have a phase lead synchronization effect in which the phase φ_2 of the response system is Δ_0 ahead of the drive's one:

$$\varphi_2 = \varphi_1 + \Delta_0. \tag{4}$$

Below we show that a similar phenomenon can appear in unidirectionally coupled chaotic Rössler systems, provided the response system is faster than the drive.

3 Phase lead synchronization in Rössler systems

We consider two unidirectionally coupled Rössler systems with different characteristic frequencies. The first system is a drive system

$$\dot{x}_1 = -\omega_1 y_1 - z_1, \tag{5a}$$
$$\dot{y}_1 = \omega_1 x_1 + a y_1, \tag{5b}$$
$$\dot{z}_1 = b + z_1(x_1 - c) \tag{5c}$$

and the second system is a response system

$$\dot{x}_2 = -\omega_2 y_2 - z_2 + \varepsilon(x_1 - x_2), \tag{6a}$$
$$\dot{y}_2 = \omega_2 x_2 + a y_2, \tag{6b}$$
$$\dot{z}_2 = b + z_2(x_2 - c). \tag{6c}$$

System's parameters are the same as in [Rosenblum, Pikovsky and Kurths, 1997]: $a = 0.165$, $b = 0.2$, $c = 10$, $\omega_1 = 0.95$, $\omega_2 = 0.99$. The parameters $\omega_{1,2}$ and ε represent the characteristic frequencies of chaotic oscillators and the coupling strength, respectively. The frequency of the slave system is slightly larger than that of the master.

To describe the phase lead synchronization we define the phases and amplitudes of chaotic oscillations in the same way as in [Rosenblum, Pikovsky and Kurths, 1997]:

$$\varphi_{1,2} = \arctan\frac{y_{1,2}}{x_{1,2}}, \quad A_{1,2} = (x_{1,2}^2 + y_{1,2}^2)^{1/2}. \tag{7}$$

To characterize different synchronization regimes that appear in system (5)-(6) with the increase of the coupling strength ε we have analyzed numerically various characteristics, namely, the difference between averaged frequencies

$$\Delta\Omega = \langle \dot{\varphi}_2 - \dot{\varphi}_1 \rangle, \tag{8}$$

the similarity function

$$S^2(\tau) = \frac{< [x_2(t) - x_1(t+\tau)]^2 >}{[< x_1^2(t) >< x_2^2(t) >]^{1/2}}, \tag{9}$$

and the conditional Lyapunov exponents $\lambda_1 > \lambda_2 > \lambda_3$ of the response system determined by the variational equations

$$\delta\dot{x}_2 = -\omega_2 \delta y_2 - \delta z_2 - \varepsilon \delta x_2, \tag{10a}$$
$$\delta\dot{y}_2 = \omega_2 \delta x_2 + a\delta y_2, \tag{10b}$$
$$\delta\dot{z}_2 = z_2 \delta x_2 + (x_2 - c)\delta z_2. \tag{10c}$$

The effect of the phase lead synchronization has been numerically detected from the dependencies of the averaged frequency difference $\Delta\Omega$, the similarity function $S(\tau)$ and three conditional Lyapunov exponents $\lambda_{1,2,3}$ on the coupling strength ε. The transition to the phase locking at some $\varepsilon = \varepsilon_l$ is characterized by vanishing of the averaged frequency difference $\Delta\Omega$. At the same coupling strength, the largest conditional Lyapunov exponent λ_1 becomes negative. In the phase locking regime, the phase difference $\varphi_2 - \varphi_1$ is positive and oscillates in a narrow interval around some mean value Θ_0. This confirms the effect of the phase lead synchronization. When the phase lead synchronization takes place, the similarly function $S(\tau)$ has a deep minimum at some fixed value $\tau = \tau_0 > 0$. The further increase of the coupling strength leads to considerable decrease of the minimum of the similarity function such that at some $\varepsilon = \varepsilon_a$ one can speak about a transition to the anticipating synchronization in which the response system anticipates the dynamics of the drive system, $x_2(t) \approx x_1(t + \tau_0)$. The anticipating time τ_0 depends on the coupling strength and decreases with the increase of ε.

The numerical results have been confirmed analytically. The analytical treatment has been performed by rewriting the model Eqs. (5)-(6) in the polar coordinates (7). Assuming that the amplitudes vary slowly and averaging the equations over rotations of the phases $\varphi_{1,2}$ we have derived an approximate dynamical equation for the phase difference $\theta = \varphi_2 - \varphi_1$:

$$\dot{\theta} = \omega_2 - \omega_1 - \frac{\varepsilon}{2}\frac{A_1}{A_2}\sin\theta. \qquad (11)$$

This equation is similar to Eq. (2) considered in the simple model. It has a positive stable fixed point

$$\Theta_0 = \arcsin\left[\frac{2A_2(\omega_2 - \omega_1)}{A_1\varepsilon}\right] > 0 \qquad (12)$$

provided $\varepsilon > 2A_2(\omega_2 - \omega_1)/A_1$. If this condition is fulfilled we have the phase lead synchronization at which the phase of the response system is Θ_0 ahead of the drive system, $\varphi_2 = \varphi_1 + \Theta_0$. Assuming $A_1 \approx A_2$ the transition point to the phase lead synchronization can be just estimated as $\varepsilon_p \approx 2(\omega_2 - \omega_1) = 0.08$. This is in good agreement with the numerical results obtained from computation of the difference between mean frequencies $\Delta\Omega$ as well as the maximal conditional Lyapunov exponent λ_1.

4 Conclusion

A new phenomenon of phase lead synchronization has been detected and analyzed numerically and analytically in unidirectionally coupled chaotic Rössler systems. If the response system is faster than the drive system than for sufficiently large coupling strength the phases are locked in such a way that the phase of the response system is ahead of the drive. This phenomenon can lead to anticipating synchronization, which does not require any delay terms either in the drive or response systems.

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