

Delayed feedback control of the Lorenz system: An analytical treatment at a subcritical Hopf bifurcation

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We develop an analytical approach for the delayed feedback control of the Lorenz system close to a subcritical Hopf bifurcation. The periodic orbits arising at this bifurcation have no torsion and cannot be stabilized by a conventional delayed feedback control technique. We utilize a modification based on an unstable delayed feedback controller. The analytical approach employs the center manifold theory and the near identity transformation. We derive the characteristic equation for the Floquet exponents of the controlled orbit in an analytical form and obtain simple expressions for the threshold of stability as well as for an optimal value of the control gain. The analytical results are supported by numerical analysis of the original system of nonlinear differential-difference equations.

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I. INTRODUCTION

One of the most popular methods in chaos control research is the delayed feedback control (DFC) method [1]. The method allows a noninvasive stabilization of unstable periodic orbits (UPOs) of dynamical systems. To apply this method no exact knowledge of either the form of the periodic orbit or the system equations is needed. The delayed feedback control algorithm has been implemented successfully in experiments as diverse as electronic chaotic oscillators [2], mechanical pendulums [3], lasers [4], gas discharge systems [5], a current-driven ion acoustic instability [6], a chaotic Taylor-Couette flow [7], chemical systems [8], high-power ferromagnetic resonance [9], helicopter rotor blades [10], and a cardiac system [11]. In the literature, many interesting suggestions have been put forward for further application of the method (see Ref. [12] for a review). Recently a challenging idea has been proposed [13,14] to use DFC for controlling pathological brain rhythms.

The DFC method is based on the online measurement of a single output signal $s(t)$ that is a function of the current system state $x(t)$, $s(t)=g(x(t))$, and uses the time-delayed difference $s(t)-s(t-\tau)$ multiplied by a factor K as a control signal. If the delay time τ is equal to the period T of an unstable periodic orbit of the system, the orbit may become stable under the appropriate choice of feedback strength K . The method is noninvasive in the sense that the control force $K[s(t)-s(t-\tau)]$ vanishes when the target state is reached.

Although the method is popular in experimental investigations, its theory is still in infancy. Systems with time delay are hard to handle because the dynamics takes place in infinite-dimensional phase spaces. Even linear analysis of such systems is difficult due to the infinite number of Floquet exponents characterizing the stability of controlled orbits.

The linear and nonlinear analysis of such systems is usually performed numerically. So far, the only analytical result of a general character has been obtained. It has been proven that the method cannot stabilize UPOs with an odd number of real positive Floquet exponents (the odd number limitation) [15,16]. This is a topological limitation related to the absence of a torsion of the controlled UPO. In addition, there are some analytical results concerning quantitative estimation of the stability of UPOs subjected to delayed feedback [16,17], but they are of limited generality. Such estimations are elaborated only for UPOs arising from a flip bifurcation.

In this context, a reasonable way for further development of the delayed feedback control theory is to look for problems allowing an analytical treatment. Our idea for the analytical approach is to consider dynamical systems close to bifurcation points of periodic orbits. Some advances in this direction have been recently achieved for dynamical systems close to the subcritical Hopf [18] as well as Neimark-Sacker (discrete Hopf) [19] bifurcations.

In a short letter [20], we have recently proposed a modified delayed feedback controller with an additional unstable mode in order to overcome the odd number limitation. The success of such a modification has been numerically demonstrated for the Lorenz system but no theoretical foundation has been presented. In this paper, we extend the ideas of the letter [20]. We develop a systematic analytical approach for delayed feedback control of dynamical systems close to a subcritical Hopf bifurcation. UPOs arising from this bifurcation have no torsion and cannot be stabilized by the conventional DFC technique. We demonstrate our approach for the Lorenz system as a representative of dynamical systems with torsion free unstable periodic orbits. Note that the control of a simple second-order dynamical system close to a subcritical Hopf bifurcation has been considered in Ref. [18]. However, the theory presented in Ref. [18] cannot be applied for high-dimensional systems. An analytical approach developed in this paper is applicable for any dynamical system with an arbitrary large phase space dimension.

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The rest of the paper is organized as follows. In Sec. II, we formulate the problem and introduce the control algorithm. Section III is devoted to the analysis of the free Lorenz system. By using the center manifold theory and near identity transformation we obtain an analytical solution for an unstable limit cycle arising in the neighborhood of the subcritical Hopf bifurcation. Then similar analysis is performed for the controlled Lorenz system in Sec. IV. In the end of this section, we analyze numerically the original system under delayed feedback control and confirm the validity of the analytical results. The paper is finished by conclusions presented in Sec. V.

II. PROBLEM FORMULATION

We consider the paradigmatic chaotic system

$$\dot{x} = \sigma(y - x), \quad (1a)$$

$$\dot{y} = rx - y - xz, \quad (1b)$$

$$\dot{z} = xy - bz \quad (1c)$$

originally introduced by Lorenz [21] as a model of turbulent convection. In usual considerations of this system the parameters σ and b are fixed, respectively, to the values 10 and 8/3, and analysis is performed for the variable parameter r . For $0 < r < 1$, the Lorenz system has a unique stable steady state (a stable node) at the origin $C^0: (0, 0, 0)$. For $r > 1$, the origin becomes a saddle and two additional symmetrical stable fixed points C^\pm ,

$$(x_f^\pm, y_f^\pm, z_f^\pm) = (\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1), \quad (2)$$

appear. For $r > r_H$, the steady states C^\pm become unstable. The value

$$r_H = \frac{\sigma(\sigma + b + 3)}{\sigma - b - 1} \approx 24.736\ 842\ 11 \quad (3)$$

represents the point at which the subcritical Hopf bifurcation occurs. Just below this bifurcation point, for

$$r = r_H - \Delta r, \quad 0 < \Delta r \ll r_H, \quad (4)$$

there are two small unstable limit cycles surrounding the stable steady states C^\pm . Moreover, at the same values of the parameter r there exists a strange attractor. Thus the system is multistable and depending on initial conditions the phase trajectory may either be attracted to the one of the steady states or exhibit a chaotic behavior on the strange attractor.

Our aim is to stabilize the unstable limit cycles arising at the Hopf bifurcation using the delayed feedback control technique. In particular, we are interested in analytical treatment of this problem. Note that the periodic orbits arising at this bifurcation are torsion-free and we need an unstable controller. Specifically, we consider the following control algorithm:

$$\dot{x} = \sigma(y - x), \quad (5a)$$

$$\dot{y} = rx - y - xz + W(y - y_f), \quad (5b)$$

$$\dot{z} = xy - bz, \quad (5c)$$

$$\dot{W} = \Lambda_c W + K[y - y(t - \tau)]. \quad (5d)$$

Here as well as in Ref. [20] we suppose that y is an observable and apply the control perturbation $W(y - y_f)$ only to the second equation of the Lorenz system. However, in contrast to Ref. [20] we use a nonlinear perturbation. As pointed out in Ref. [18] this is a necessary requirement when considering the system close to a Hopf bifurcation in order to provide the coupling between the controlled system and controller in averaged equations. The parameter y_f in the perturbation is the value of the observable when the system is in one of the stable steady states C^\pm . For definiteness, we consider the control of the periodic orbit surrounding the fixed point C^+ and take $y_f \equiv y_f^+ = \sqrt{b(r-1)}$. Note that the value y_f can be measured experimentally, since C^+ is the stable fixed point.

Equation (5d) describes an unstable delayed feedback controller, which supplements the system with an additional unstable Floquet mode and eliminates the odd number limitation. The positive parameter $\Lambda_c > 0$ defines the value of the additional Floquet exponent. The parameter K denotes the strength of the feedback gain. The delay time τ in Eq. (5d) is equal to the period T of the unstable periodic orbit such that the controller does not change the periodic solutions of the Lorenz system with the period $T = \tau$. Thus if the stabilization of the periodic orbit is successful there is no power dissipated in the feedback loop.

In a real experiment, the period T of an UPO is not known *a priori* and different strategies can be used for selection of the right value of the delay time τ . A universal approach is based on minimization of the amplitude of the feedback perturbation $K[y - y(t - \tau)]$ [1]. An adaptive technique with automatic adjustment of the delay time has been considered in Ref. [22]. Another approach is based on minimization of the difference $\tau - T_y(\tau)$, where $T_y(\tau)$ is the period of the output signal $y(t)$, which generally differs from T if $\tau \neq T$ [23].

III. ANALYSIS OF THE FREE LORENZ SYSTEM

We start our analysis with the free Lorenz system (1). First we transform the variables using the eigenvectors of the steady state C^+ at the bifurcation point $r = r_H$ as a basis for a new coordinate system. Then applying the center manifold theory we eliminate the fast nonoscillating mode and obtain a reduced system for oscillating modes. Using the near identity transformation we transform the equations for the oscillating modes to the normal form of the subcritical Hopf bifurcation. As a final result of this section, we obtain an analytical solution for the unstable periodic orbit arising from this bifurcation.

A. Transforming the system variables

Our aim is the control of the unstable limit cycle surrounding the stable fixed point C^+ . Thus it is convenient to

shift the origin to this point by using the transformation

$$x = x_f^+ + u_1, \quad y = y_f^+ + u_2, \quad z = z_f + u_3. \quad (6)$$

Defining the state vector

$$A_0 = \begin{pmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & -\sqrt{b(r_H-1)} \\ \sqrt{b(r_H-1)} & \sqrt{b(r_H-1)} & -b \end{pmatrix} = \begin{pmatrix} -10 & 10 & 0 \\ 1 & -1 & -\frac{2}{57}\sqrt{51\,414} \\ \frac{2}{57}\sqrt{51\,414} & \frac{2}{57}\sqrt{51\,414} & -\frac{8}{3} \end{pmatrix}, \quad (9)$$

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 1 & 1 & 0 \end{pmatrix}, \quad N(\mathbf{u}) = \begin{pmatrix} 0 \\ -u_1 u_3 \\ u_1 u_2 \end{pmatrix}, \quad (10)$$

and the parameter

$$\varepsilon = \sqrt{b(r_H-1)} - \sqrt{b(r-1)} \approx \sqrt{b/(r_H-1)} \Delta r / 2 \quad (11)$$

defines the closeness of the system to the bifurcation point $r=r_H$. This is the main control parameter, whose smallness we exploit in the following perturbation theory.

The first two terms $A_0 \mathbf{u}$ and $\varepsilon A_1 \mathbf{u}$ in the right-hand side of Eq. (8) represent the linear part of the vector field, while the last term $N(\mathbf{u})$ defines the nonlinear part. The matrix $A_0 - \varepsilon A_1$ is the Jacobian derivative at the fixed point C^+ , where A_0 is the value of the Jacobian calculated at the bifurcation point $r=r_H$ and εA_1 is a small deviation due to the shift of the parameter r from the bifurcation point.

We now transform the system variables in such a way as to diagonalize the unperturbed linear part $A_0 \mathbf{u}$ of the vector field. With this aim, we solve the eigenvalue problem for the matrix A_0 ,

$$A_0 \boldsymbol{\phi}^{(\mu)} = \gamma_\mu \boldsymbol{\phi}^{(\mu)}. \quad (12)$$

As a result we obtain three eigenvalues

$$\gamma_1 = \gamma_2^* \equiv i\omega \approx 9.624\,530\,060i, \quad \gamma_3 \approx -13.666\,666\,63 \quad (13)$$

and three corresponding eigenvectors

$$\boldsymbol{\phi}^{(1)} \approx \begin{pmatrix} 0.286\,306\,771\,5 - 0.239\,527\,044\,9i \\ 0.516\,840\,295\,9 + 0.036\,029\,768\,49i \\ 0.014\,609\,827\,41 - 0.659\,865\,377\,3i \end{pmatrix},$$

$$\boldsymbol{\phi}^{(2)} = \boldsymbol{\phi}^{(1)*},$$

$$\boldsymbol{\phi}^{(3)} \approx \begin{pmatrix} 0.874\,089\,854\,1 \\ -0.320\,499\,613\,6 \\ -0.400\,397\,703\,1 \end{pmatrix}. \quad (14)$$

$$\mathbf{u} = (u_1 \quad u_2 \quad u_3)^T, \quad (7)$$

we rewrite the Lorenz equations in the matrix form

$$\dot{\mathbf{u}} = A_0 \mathbf{u} - \varepsilon A_1 \mathbf{u} + N(\mathbf{u}), \quad (8)$$

where the matrices A_0 , A_1 , and $N(\mathbf{u})$ are

The first two eigenvalues are imaginary and their eigenvectors are complex conjugates. The third eigenvalue is real and negative. We use the eigenvectors (14) as a basis for a new coordinate system and apply a linear transformation of the system variables:

$$\mathbf{u}(t) = \sum_{\mu=1}^3 \xi_\mu(t) \boldsymbol{\phi}^{(\mu)}. \quad (15)$$

The new dynamic variables $\xi_\mu(t)$ ($\mu=1,2,3$) define the amplitudes of the system eigenmodes at the bifurcation point $r=r_H$. Note that $\xi_1(t)$ and $\xi_2(t)$ have to be complex conjugate, $\xi_2^*(t) = \xi_1(t)$, in order to provide the real valued solution for $\mathbf{u}(t)$. To derive equations for the new variables $\xi_\mu(t)$ we insert Eq. (15) in system (8):

$$\sum_{\mu=1}^3 \dot{\xi}_\mu \boldsymbol{\phi}^{(\mu)} = \sum_{\mu=1}^3 \lambda_\mu \xi_\mu \boldsymbol{\phi}^{(\mu)} - \varepsilon \sum_{\mu=1}^3 A_1 \boldsymbol{\phi}^{(\mu)} \xi_\mu + N\left(\sum_{\mu=1}^3 \xi_\mu \boldsymbol{\phi}^{(\mu)}\right). \quad (16)$$

To obtain equations for $\xi_\mu(t)$ in the explicit form one needs the adjoint eigenvalue equation

$$\boldsymbol{\psi}^{(v)} A_0 = \gamma_v \boldsymbol{\psi}^{(v)}. \quad (17)$$

The solutions of this equation satisfying the normalization conditions

$$\langle \boldsymbol{\psi}^{(v)} | \boldsymbol{\phi}^{(\mu)} \rangle = \delta_{v\mu}, \quad \mu, v = 1, 2, 3, \quad (18)$$

are

$$\boldsymbol{\psi}^{(1)} \approx \begin{pmatrix} 0.274\,371\,821\,3 + 0.245\,644\,438\,8i \\ 0.816\,981\,259\,6 - 0.154\,736\,036\,5i \\ -0.054\,986\,710\,09 + 0.660\,114\,055\,5i \end{pmatrix}^T,$$

$$\boldsymbol{\psi}^{(2)} = \boldsymbol{\psi}^{(1)*},$$

$$\boldsymbol{\psi}^{(3)} \approx \begin{pmatrix} 0.829\ 679\ 010\ 8 \\ -0.450\ 397\ 177\ 3 \\ -0.325\ 760\ 791\ 5 \end{pmatrix}^T. \quad (19)$$

Multiplying Eq. (16) by $\boldsymbol{\psi}^{(\nu)}$ from the left side and using conditions (18) one obtains finally the equations for the eigenmodes,

$$\begin{aligned} \dot{\xi}_\nu &= \gamma_\nu \xi_\nu - \varepsilon \sum_{\mu=1}^3 \langle \boldsymbol{\psi}^{(\nu)} | A_1 | \boldsymbol{\phi}^{(\mu)} \rangle \xi_\mu + g_\nu(\xi_1, \xi_2, \xi_3) \\ &\equiv f_\nu(\xi_1, \xi_2, \xi_3), \quad \nu = 1, 2, 3, \end{aligned} \quad (20)$$

where $g_\nu(\xi_1, \xi_2, \xi_3)$ are the nonlinear functions

$$g_\nu(\xi_1, \xi_2, \xi_3) = \left\langle \boldsymbol{\psi}^{(\nu)} \left| N \left(\sum_{\mu=1}^3 \xi_\mu \boldsymbol{\phi}^{(\mu)} \right) \right. \right\rangle, \quad \nu = 1, 2, 3. \quad (21)$$

Until now Eqs. (20) are exact. They are equivalent to the original Lorenz system (1). However, this form is more convenient when analyzing the system dynamics close to the Hopf bifurcation, for small values of the parameter ε .

B. Reducing the system dimension

Close to the bifurcation point $\varepsilon=0$, Eqs. (20) admit an analytical treatment. First we note that for $\varepsilon=0$, the linear part of the vector field is diagonal and small deviations from the origin are described by three linear independent modes $\dot{\xi}_\nu = \gamma_\nu \xi_\nu$, $\nu=1,2,3$. The first two modes are oscillating, $\gamma_{1,2} = \pm i\omega$, and the third mode is decaying, $\gamma_3 < 0$. This enables us to apply the center manifold theory and exclude the decaying mode.

To get a reduced system of equations for the oscillating modes in relation to the parameter ε , we change for a time the role of the parameter ε . We regard ε as an additional dependent variable that satisfies the trivial equation

$$\dot{\varepsilon} = 0. \quad (22)$$

Then for the extended system (20), (22), linearized at the fixed point $(\xi_1, \xi_2, \xi_3, \varepsilon) = (0, 0, 0, 0)$, the ξ_3 axis is a stable subspace and $(\xi_1, \xi_2, \varepsilon)$ is the center subspace. Thus according to the well known theorem, in the $(\xi_1, \xi_2, \xi_3, \varepsilon)$ phase space there exists a center manifold

$$\xi_3 = h(\xi_1, \xi_2, \varepsilon) \quad (23)$$

tangent to the center subspace

$$h(0, 0, 0) = 0, \quad \left. \frac{\partial h}{\partial \xi_1} \right|_{(0,0,0)} = \left. \frac{\partial h}{\partial \xi_2} \right|_{(0,0,0)} = \left. \frac{\partial h}{\partial \varepsilon} \right|_{(0,0,0)} = 0. \quad (24)$$

We expand the manifold function in Taylor series up to the second-order terms

$$\begin{aligned} h(\xi_1, \xi_2, \varepsilon) &= K_{200} \xi_1^2 + K_{020} \xi_2^2 + K_{002} \varepsilon^2 + K_{011} \xi_2 \varepsilon + K_{101} \xi_1 \varepsilon \\ &\quad + K_{110} \xi_1 \xi_2. \end{aligned} \quad (25)$$

The linear terms are omitted to satisfy the conditions (24). Differentiating (23) we obtain

$$\dot{\xi}_3 = \frac{\partial h}{\partial \xi_1} \dot{\xi}_1 + \frac{\partial h}{\partial \xi_2} \dot{\xi}_2; \quad (26)$$

in what follows

$$\begin{aligned} f_3(\xi_1, \xi_2, h(\xi_1, \xi_2, \varepsilon)) &= \frac{\partial h}{\partial \xi_1} f_1(\xi_1, \xi_2, h(\xi_1, \xi_2, \varepsilon)) \\ &\quad + \frac{\partial h}{\partial \xi_2} f_2(\xi_1, \xi_2, h(\xi_1, \xi_2, \varepsilon)). \end{aligned} \quad (27)$$

By equating coefficients at different orders $\xi_1^n \xi_2^m \varepsilon^l$ ($n+m+l=2$) we get a linear system of equations yielding the coefficients of the expansion (25). Solving this system we obtain

$$K_{200} \approx -0.004\ 667\ 161\ 467 + 0.002\ 937\ 029\ 495i,$$

$$K_{020} = K_{200}^*, \quad K_{002} = 0,$$

$$K_{011} \approx 0.020\ 429\ 285\ 22 - 0.002\ 508\ 863\ 404i,$$

$$K_{101} = K_{011}^*, \quad K_{110} \approx 0.004\ 050\ 541\ 094. \quad (28)$$

Substituting Eq. (23) in the two first equations of the system (20) we obtain two equations for the oscillating modes,

$$\dot{\xi}_\nu = f_\nu(\xi_1, \xi_2, h(\xi_1, \xi_2, \varepsilon)), \quad \nu = 1, 2. \quad (29)$$

Due to the property $\xi_2 = \xi_1^*$ these equations are equivalent. Using the notations

$$\xi_1 = \xi, \quad \xi_2 = \xi^* = \xi^* \quad (30)$$

they can be presented in the form

$$\dot{\xi} = f_1(\xi, \xi^*, h(\xi, \xi^*, \varepsilon)). \quad (31)$$

The function $f_1(\xi, \xi^*, h(\xi, \xi^*, \varepsilon))$ in Eq. (31) is rather complicated. It contains terms up to fourth order. However, most of them drop out when transforming this equation to the normal form. We write out explicitly only the relevant terms:

$$\dot{\xi} = i\omega \xi + \varepsilon a_{10} \xi + a_{21} |\xi|^2 \xi + a_{20} \xi^2 + a_{02} \xi^{*2} + a_{11} |\xi|^2 + \dots \quad (32)$$

Here a_{10} , a_{21} , a_{20} , a_{02} , and a_{11} are the complex constants

$$a_{10} \approx -0.180\ 337\ 995\ 4 - 1.082\ 716\ 628i,$$

$$a_{21} \approx 0.000\ 471\ 257\ 092\ 6 + 0.003\ 413\ 821\ 462i,$$

$$a_{20} \approx 0.221\ 785\ 599\ 3 + 0.243\ 013\ 769\ 5i,$$

$$a_{02} \approx 0.012\ 414\ 258\ 38 - 0.083\ 878\ 778\ 90i,$$

$$a_{11} \approx -0.280\,415\,843\,5 + 0.234\,175\,169\,6i. \quad (33)$$

Equation (32) represents the reduced system, which describes well the Lorenz dynamics close to the subcritical Hopf bifurcation. To obtain the solution for the unstable limit cycle arising from this bifurcation, in the next section we transform this equation to the normal form.

C. Near identity transformation and parameters of the unstable limit cycle

The reduced system (32) can be transformed to the normal form of the subcritical Hopf bifurcation

$$\dot{\eta} = \lambda_f \eta + c_1 \eta^2 \eta^* + O(|\eta|^5) \quad (34)$$

either by using a perturbation theory based on the multiscale expansion or by applying a near identity transformation. Both approaches lead to the same result, but the second approach is simpler to handle and we use it in this paper.

Following Ref. [24], the near identity transformation that transforms Eq. (32) to the form (34) is

$$\xi = \eta + \sum_{2 \leq j+k \leq 3} \kappa_{jk} \frac{\eta^j \eta^{*k}}{j! k!} + O(|\eta|^4). \quad (35)$$

Substituting Eq. (35) in (32) and using Eq. (34) one obtains a polynomial equation with respect to η and η^* . Equating coefficients of the polynomial equation yields the parameters λ_f and c_1 of the normal form (34),

$$\lambda_f = i\omega + \varepsilon a_{10}, \quad (36)$$

$$c_1 = \frac{i}{\omega} \left(a_{20} a_{11} - \frac{2}{3} a_{02} a_{02}^* - a_{11} a_{11}^* \right) + a_{21} + O(\varepsilon) \\ \approx 0.002\,155\,311\,349 - 0.023\,326\,623\,61i, \quad (37)$$

and the coefficients κ_{jk} of the near identity transformation (35). The coefficients of leading terms in the transformation are

$$\kappa_{20} = \frac{2a_{20}}{\lambda_f}, \quad \kappa_{11} = \frac{a_{11}}{\lambda_f^*}, \quad \kappa_{02} = \frac{a_{02}}{2\lambda_f^* - \lambda_f}. \quad (38)$$

Equation (36) defines the eigenvalue of the fixed point C^+ . For $\varepsilon > 0$, it is stable since $\text{Re}(\lambda_f) = \varepsilon \text{Re}(a_{10}) < 0$. The unstable limit cycle surrounding this fixed point can be found by solving the normal form equation (34). By substitution $\eta = R \exp(i\Theta)$, where R and Θ are real-valued variables, this equation can be presented in the form

$$\dot{\Theta} = \text{Im}(\lambda_f) + \text{Im}(c_1) R^2, \quad (39a)$$

$$\dot{R} = [\text{Re}(\lambda_f) + \text{Re}(c_1) R^2] R. \quad (39b)$$

From Eqs. (39b) and (36) it follows that the radius of the limit cycle is

$$R_0 = \sqrt{-\frac{\text{Re}(a_{10})}{\text{Re}(c_1)}} \sqrt{\varepsilon} \approx 9.147\,209\,186 \sqrt{\varepsilon}. \quad (40)$$

The frequency of the limit cycle is determined by the right-hand side of Eq. (39a) at $R=R_0$:

$$\omega_0 = \omega + \left(\text{Im}(a_{10}) - \frac{\text{Im}(c_1)}{\text{Re}(c_1)} \text{Re}(a_{10}) \right) \varepsilon \\ \approx 9.624\,530\,060 - 3.034\,488\,720 \varepsilon. \quad (41)$$

Thus the analytical solution of the normal form equation (34) corresponding to the unstable limit cycle is

$$\eta_0(t) = R_0 \exp(i\omega_0 t). \quad (42)$$

The dynamics of the mode ξ corresponding to this solution is obtained by inserting Eq. (42) in the transformation formula (35).

Finally, one can easily obtain the Floquet exponent Λ_0 of the unstable limit cycle. Linearization of Eq. (39b) at the fixed point $R=R_0$ leads to the expression

$$\Lambda_0 = \text{Re}(\lambda_f) + 3\text{Re}(c_1) R_0^2 = -2 \text{Re}(a_{10}) \varepsilon \approx 0.360\,675\,990\,7 \varepsilon. \quad (43)$$

We see that the the Floquet exponent is proportional to the parameter ε and is positive for $\varepsilon > 0$.

IV. ANALYSIS OF THE CONTROLLED LORENZ SYSTEM

In this section, we analyze the Lorenz system under delayed feedback control described by Eqs. (5). We exploit the relationship between the characteristic equations for the Floquet exponents of two different control problems, the delayed feedback control and the proportional feedback control. The latter problem is much simpler and admits an analytical treatment similar to that described in Sec. III. As a result we obtain the characteristic equations for the Floquet exponents of the Lorenz system under delayed feedback control in analytical form. We finish the section with numerical analysis of the original system of nonlinear differential-difference Eqs. (5).

A. Proportional versus delayed feedback

First, we rescale the controller variable and parameters

$$W = \varepsilon w, \quad \Lambda_c = \varepsilon \lambda_c, \quad K = \varepsilon k \quad (44)$$

to rewrite the controlled Lorenz system (5) in a more convenient form

$$\dot{x} = \sigma(y - x), \quad (45a)$$

$$\dot{y} = rx - y - xz + \varepsilon w(y - y_f), \quad (45b)$$

$$\dot{z} = xy - bz, \quad (45c)$$

$$\dot{w} = \varepsilon \lambda_c w + k[y - y(t - \tau)]. \quad (45d)$$

Generally this is a rather complicated system of nonlinear differential-difference equations. The dynamics of the system takes place in an infinite-dimensional phase space and reduction of the phase space dimension via the center manifold theory is a nontrivial task. To overcome the problem of an infinite-dimensional phase space we proceed in the following way.

Analogously to Ref. [17], we consider the proportional feedback control instead of the delayed feedback control, i.e., in Eq. (45d) we replace the delay term $y(t-\tau)$ with the periodic solution of the free Lorenz system $y_0(t)$ corresponding to the unstable limit cycle, which we intend to stabilize. Then in place of the system (45) we get

$$\dot{x} = \sigma(y - x), \quad (46a)$$

$$\dot{y} = rx - y - xz + \varepsilon w(y - y_f), \quad (46b)$$

$$\dot{z} = xy - bz, \quad (46c)$$

$$\dot{w} = \varepsilon \lambda_c w + k[y - y_0(t)]. \quad (46d)$$

Both Eqs. (45) and (46) have the same periodic solution corresponding to the desired limit cycle. Although the limit cycle has different Floquet exponents for the systems (45) and (46), there exists a relationship between these two Floquet problems.

In the case of the delayed feedback control, the Floquet exponents are determined by linearization of the system (45):

$$\delta \dot{x} = \sigma(\delta y - \delta x), \quad (47a)$$

$$\delta \dot{y} = (r - z_0)\delta x - \delta y - x_0 \delta z + \varepsilon(y_0 - y_f)\delta w, \quad (47b)$$

$$\delta \dot{z} = -b\delta z + x_0 \delta y + y_0 \delta x, \quad (47c)$$

$$\delta \dot{w} = \varepsilon \lambda_c \delta w + k[\delta y - \delta y(t - \tau)]. \quad (47d)$$

Here $[\delta x, \delta y, \delta z]$ denote small deviations from the periodic orbit $[x_0(t), y_0(t), z_0(t)] = [x_0(t + \tau), y_0(t + \tau), z_0(t + \tau)]$ that satisfies the free system (1), and $\delta w = w$. Due to the Floquet theory the delay term $\delta y(t - \tau)$ in Eq. (47d) can be eliminated and the system of differential-difference equations (47) can be transformed to a system of ordinary differential equations. The Floquet decomposition of solutions of the system (47) implies that $\delta y(t) = \exp(\Lambda t)U(t)$, where Λ is the Floquet exponent and $U(t) = U(t - \tau)$ is a periodic function. It follows that the delay term can be expressed as $\delta y(t - \tau) = \exp(-\Lambda \tau)\delta y(t)$. The price one has to pay for the elimination of the delay term is that the variational equations (47) defining the Floquet exponent depend on the Floquet exponent itself.

In the case of the proportional feedback control the Floquet exponents are defined by linearization of the system (46). This leads to variational equations similar to Eqs. (47) with the only difference that the last term in Eq. (47d), $k[\delta y - \delta y(t - \tau)] = k[1 - \exp(-\Lambda \tau)]\delta y$, is replaced by $k\delta y$. It follows that the Floquet exponents for the delayed feedback control can be obtained from linearized system (46) by using the substitution

$$k \rightarrow k[1 - \exp(-\Lambda \tau)]. \quad (48)$$

Thus rather than analyzing the system of differential-difference Eqs. (45) we can focus on the analysis of the simpler system (46) described by ordinary differential equa-

tions. If we manage to derive analytically a characteristic equation for the Floquet exponents of the system (46), then we can use the substitution (48) and obtain the characteristic equation for the case of the delayed feedback control system (45).

The system (46) is nonautonomous due to the time-dependent term $y_0(t)$ in Eq. (46d). It is convenient to transform this system to the autonomous form by supplementing it with an additional free Lorenz system

$$\dot{x}_0 = \sigma(y_0 - x_0), \quad (49a)$$

$$\dot{y}_0 = rx_0 - y_0 - x_0 z_0, \quad (49b)$$

$$\dot{z}_0 = x_0 y_0 - bz_0. \quad (49c)$$

We suppose that the initial conditions of the system (49) are chosen on the stable manifold of the desired limit cycle, such that its solution converges to the limit cycle. As a result these equations generate the periodic signal $y_0(t) = y_0(t + \tau)$ which is used as an input in Eq. (46d).

Our aim now is to reduce the dimension of the system (46), (49). But first we transform the variables in a similar way as it has been done in Sec. III A. We shift the origin in Eqs. (46) and (49) to the fixed point C^+ and obtain

$$\dot{\mathbf{u}} = A_0 \mathbf{u} - \varepsilon A_1 \mathbf{u} + (0 \quad 1 \quad 0)^T \varepsilon w u_2 + \mathbf{N}(\mathbf{u}), \quad (50a)$$

$$\dot{w} = \varepsilon \lambda_c w + k(u_2 - u_{02}), \quad (50b)$$

$$\dot{\mathbf{u}}_0 = A_0 \mathbf{u}_0 - \varepsilon A_1 \mathbf{u}_0 + \mathbf{N}(\mathbf{u}_0). \quad (50c)$$

Here the vector \mathbf{u}_0 is defined analogously to the vector \mathbf{u} in Eqs. (6) and (7),

$$\mathbf{u}_0 = (u_{01} \quad u_{02} \quad u_{03})^T = (x_0 - x_f^+ \quad y_0 - y_f^+ \quad z_0 - z_f^+)^T. \quad (51)$$

Using the linear transformation (15) for the vector \mathbf{u} and the similar transformation

$$\mathbf{u}_0(t) = \sum_{\mu=1}^3 \xi_{0\mu}(t) \boldsymbol{\phi}^{(\mu)} \quad (52)$$

for the vector \mathbf{u}_0 we finally transform the Lorenz system under proportional feedback control to the form

$$\begin{aligned} \dot{\xi}_\nu &= \gamma_\nu \xi_\nu - \varepsilon \sum_{\mu=1}^3 \langle \boldsymbol{\psi}^{(\nu)} | A_1 | \boldsymbol{\phi}^{(\mu)} \rangle \xi_\mu + \varepsilon w \sum_{\mu=1}^3 \boldsymbol{\psi}_2^{(\nu)} \boldsymbol{\phi}_2^{(\mu)} \xi_\mu \\ &+ g_\nu(\xi_1, \xi_2, \xi_3) \equiv F_\nu(\xi_1, \xi_2, \xi_3, w), \quad \nu = 1, 2, 3, \end{aligned} \quad (53a)$$

$$\dot{w} = \varepsilon \lambda_c w + k \sum_{\mu=1}^3 \boldsymbol{\phi}_2^{(\mu)} (\xi_\mu - \xi_{0\mu}), \quad (53b)$$

$$\dot{\xi}_{0\nu} = \gamma_\nu \xi_{0\nu} - \varepsilon \sum_{\mu=1}^3 \langle \psi^{(\nu)} | A_1 | \phi^{(\mu)} \rangle \xi_{0\mu} + g_\nu(\xi_1, \xi_2, \xi_3), \quad (53c)$$

$$\nu = 1, 2, 3,$$

convenient for application of the center manifold theory. The nonlinear functions g_ν are defined by Eq. (21).

For small value of the parameters ε , λ_c , and k , the system (53) can be treated analytically in much the same way as the free Lorenz system.

B. Reducing the dimension of the Lorenz system controlled by proportional feedback

First we reduce the dimension of the system (53) using the technique described in Sec. III B. For $\varepsilon=0$, $\lambda_c=0$, and $k=0$, the linear part of the vector field is diagonal and small deviations from the origin are described by $\dot{\xi}_\nu = \gamma_\nu \xi_\nu$, $\dot{w}=0$, and $\dot{\xi}_{0\nu} = \gamma_\nu \xi_{0\nu}$ with $\nu=1, 2, 3$. The modes ξ_1 , ξ_2 , ξ_{01} , ξ_{02} are oscillating with the eigenvalues $\gamma_{1,2} = \pm i\omega$, the mode w has zero eigenvalue, and the modes ξ_3 , ξ_{03} are decaying, since $\gamma_3 < 0$. Thus one can apply the center manifold theory and exclude the decaying modes.

In order to get a reduced system of equations in relation to the parameters ε , λ_c , and k , we regard them as additional dependent variables satisfying the trivial equations

$$\dot{\varepsilon} = 0, \quad \dot{\lambda}_c = 0, \quad \dot{k} = 0. \quad (54)$$

The extended phase space of the system (53), (54) is defined by dynamical variables $(\xi_1, \xi_2, \xi_3, \varepsilon, \xi_{01}, \xi_{02}, \xi_{03}, w, k, \lambda_c)$. For this system, linearized at the origin, the plane (ξ_3, ξ_{03}) is a stable subspace and $(\xi_1, \xi_2, \xi_{01}, \xi_{02}, \varepsilon, w, k, \lambda_c)$ is the center subspace. Thus in the extended phase space there exists a center manifold

$$\begin{pmatrix} \xi_3 \\ \xi_{03} \end{pmatrix} = \begin{pmatrix} h_c(\xi_1, \xi_2, \varepsilon, \xi_{01}, \xi_{02}, w, k, \lambda_c) \\ h_0(\xi_{01}, \xi_{02}, \varepsilon) \end{pmatrix} \equiv \mathbf{H} \quad (55)$$

tangent to the center subspace at the origin:

$$\mathbf{H}(\mathbf{0}) = \mathbf{0}, \quad D\mathbf{H}(\mathbf{0}) = \mathbf{0}. \quad (56)$$

Here $D\mathbf{H}$ denotes the Jacobian of derivatives with respect to all variables $(\xi_1, \xi_2, \xi_{01}, \xi_{02}, \varepsilon, w, k, \lambda_c)$ of the center subspace. First we note that the function h_0 depends not on all variables of the center subspace but only on the subset $(\xi_{01}, \xi_{02}, \varepsilon)$. This is due to the fact that Eq. (53c) is independent of Eqs. (53a) and (53b). Equation (53c) corresponds to the free Lorenz system and coincides with Eq. (20). It follows that the manifold function h_0 coincides with that of Eq. (25), i.e., $h_0 = h(\xi_{01}, \xi_{02}, \varepsilon)$, and hence

$$\xi_{03} = h(\xi_{01}, \xi_{02}, \varepsilon). \quad (57)$$

Substituting Eq. (55) in (53) one obtains the equation for the manifold function \mathbf{H} , similar to Eq. (27). Direct analysis of this equation shows that the expansion of the function h_c up to the second-order terms also leads to the expression

(25), i.e., $h_c(\xi_1, \xi_2, \varepsilon, \xi_{01}, \xi_{02}, w, k, \lambda_c) = h(\xi_1, \xi_2, \varepsilon)$, and hence

$$\xi_3 = h(\xi_1, \xi_2, \varepsilon). \quad (58)$$

It turns out that the controller does not change the manifold function of the controlled system. Substituting Eq. (58) in the first two Eqs. (53a) one obtains the equation for the oscillating mode of the controlled system,

$$\dot{\xi} = F_1(\xi, \xi^*, h(\xi, \xi^*, \varepsilon), w), \quad (59)$$

where $\xi = \xi_1 = \xi_2^*$. Writing out explicitly only the relevant terms of the function F_1 one obtains finally the reduced equation for the controlled system in the form

$$\begin{aligned} \dot{\xi} = & i\omega\xi + \varepsilon a_{10}\xi + \varepsilon b_{11}w\xi + a_{21}|\xi|^2\xi + a_{20}\xi^2 \\ & + a_{02}\xi^{*2} + a_{11}|\xi|^2 + \dots, \end{aligned} \quad (60)$$

similar to Eq. (32). The only difference is that here we have an additional term $\varepsilon b_{11}w\xi$ related to the control perturbation. The coefficients a_{10} , a_{21} , a_{20} , a_{02} , a_{11} are defined by Eq. (33) and the value of the coefficient b_{11} is

$$b_{11} \approx 0.427\ 823\ 939\ 5 - 0.050\ 538\ 173\ 25i. \quad (61)$$

Equation (60) has to be supplemented by Eq. (53b) for the variable w to complete the system of reduced equations describing the dynamics of the Lorenz system under proportional feedback control. Writing explicitly the sum in Eq. (53b) and substituting Eqs. (57) and (58) for the decaying modes, one obtains

$$\begin{aligned} \dot{w} = & \varepsilon\lambda_c w + k\phi_2^{(1)}(\xi - \xi_0) + k\phi_2^{(2)}(\xi^* - \xi_0^*) + k\phi_2^{(3)}[h(\xi, \xi^*, \varepsilon) \\ & - h(\xi_0, \xi_0^*, \varepsilon)]. \end{aligned} \quad (62)$$

Here $\xi_0 = \xi_{01} = \xi_{02}^*$ is the solution of the free Lorenz system for the oscillating mode corresponding to the unstable limit cycle.

C. Near identity transformation and averaging

We use the near identity transformation (35) to simplify the reduced system (60), (62). First we transform Eq. (60). By substituting Eq. (35) in Eq. (60) and using the technique described in Sec. III C we derive the normal form equation similar to Eq. (34) but with an additional control perturbation $\varepsilon b_{11}w\eta$:

$$\dot{\eta} = \lambda_f \eta + c_1 \eta^2 \eta^* + \varepsilon b_{11}w\eta. \quad (63)$$

Here we have restricted ourself to the leading term in the control perturbation, i.e., in the expression $\varepsilon b_{11}w\xi = \varepsilon b_{11}w[\eta + O(|\eta|^2)]$ we have omitted the $O(|\eta|^2)$ terms.

We now simplify Eq. (62). Again we use the near identity transformation (35) and transform the variables $(\xi, \xi_0) \rightarrow (\eta, \eta_0)$. Afterward we average Eq. (62) over the period $\tau = 2\pi/\omega_0$ of the limit cycle. We suppose that w is a slowly varying variable. The variable η_0 satisfies Eq. (42), i.e., $\eta_0(t) = R_0 \exp(i\omega_0 t)$. For the variable η , we suppose that it can be presented in the form $\eta(t) = A(\varepsilon t) \exp(i\omega_0 t)$, where $A(\varepsilon t)$ is a slowly varying complex amplitude. Then the averaging

eliminates all the terms containing the fast exponents $\exp(\pm im\omega_0 t)$, with nonzero integer m , and we obtain

$$\dot{w} = \varepsilon \lambda_c w + kP(|\eta|^2 - |\eta_0|^2), \quad (64)$$

where the coefficient P is

$$P = \frac{i}{\omega} (a_{11} \phi_2^{(1)} - a_{11}^* \phi_2^{(2)}) + K_{110} \phi_2^{(3)} + O(\varepsilon) \\ \approx -0.024\,349\,264\,335. \quad (65)$$

In the next section, we exploit the simplified system of Eqs. (63), (64) to obtain the characteristic equation for the Floquet exponents of the controlled system.

D. Stability analysis of the controlled system

By the substitution $\eta = R \exp(i\Theta)$, Eqs. (63) and (64) can be presented in the form

$$\dot{\Theta} = \text{Im}(\lambda_f) + \text{Im}(c_1)R^2 + \varepsilon \text{Im}(b_{11})w, \quad (66a)$$

$$\dot{R} = [\text{Re}(\lambda_f) + \text{Re}(c_1)R^2 + \varepsilon \text{Re}(b_{11})w]R, \quad (66b)$$

$$\dot{w} = \varepsilon \lambda_c w + kP(R^2 - R_0^2). \quad (66c)$$

These equations have the solution $(\Theta, R, w) = (\omega_0 t, R_0, 0)$, which corresponds to the limit cycle of the free system. Linearization about this solution leads to the variational equations

$$\delta \dot{\Theta} = 2 \text{Im}(c_1)R_0 \delta R + \varepsilon \text{Im}(b_{11})\delta w, \quad (67a)$$

$$\delta \dot{R} = \Lambda_0 \delta R + \varepsilon \text{Re}(b_{11})R_0 \delta w, \quad (67b)$$

$$\delta \dot{w} = 2kPR_0 \delta R + \varepsilon \lambda_c \delta w. \quad (67c)$$

Here Λ_0 is the Floquet exponent of the free orbit defined by Eq. (43). Equations (67b) and (67c) are independent of Eq. (67a) and define the nonzero Floquet exponents of the controlled limit cycle, while Eq. (67a) defines the zero Floquet exponent. The nonzero Floquet exponents Λ satisfy the quadratic equation

$$\Lambda^2 - (\Lambda_0 + \varepsilon \lambda_c)\Lambda + \varepsilon \Lambda_0 \lambda_c + \varepsilon^2 Qk = 0, \quad (68)$$

where

$$Q = 2P \text{Re}(b_{11})\text{Re}(a_{10})/\text{Re}(c_1) \approx 1.743\,243\,862. \quad (69)$$

We are reminded that Eq. (68) is the characteristic equation for the Floquet exponents of the limit cycle under proportional feedback control. To derive the characteristic equation for the case of the delayed feedback control we apply the substitution (48). As a result we obtain the quasipolynomial characteristic equation

$$\Lambda^2 - (\Lambda_0 + \varepsilon \lambda_c)\Lambda + \varepsilon \Lambda_0 \lambda_c + \varepsilon^2 Qk[1 - \exp(-\Lambda \tau)] = 0. \quad (70)$$

By rescaling the Floquet exponents

$$\Lambda = \varepsilon \lambda, \quad \Lambda_0 = \varepsilon \lambda_0, \quad (71)$$

this equation can be presented in a more convenient form

$$\lambda^2 - (\lambda_0 + \lambda_c)\lambda + \lambda_0 \lambda_c + Qk[1 - \exp(-\varepsilon \lambda \tau)] = 0. \quad (72)$$

From Eqs. (43) and (71) it follows that

$$\lambda_0 \approx 0.360\,675\,990\,7. \quad (73)$$

Equation (72) is the main result of this paper. It defines the Floquet exponents of the controlled Lorenz system close to the subcritical Hopf bifurcation in relation to the bifurcation parameter ε and the parameters λ_c and k of the unstable delayed feedback controller.

The quasipolynomial Eq. (72) has an infinite number of solutions, since it defines the Floquet exponents of a system described by differential-difference equations. In the general case, the solutions of Eq. (72) can be determined numerically. However, the leading Floquet exponents close to the bifurcation point can be obtained analytically. For $\varepsilon|\lambda|\tau \ll 1$, we can use an approximation $\exp(-\varepsilon \lambda \tau) \approx 1 - \varepsilon \lambda \tau$, which transforms Eq. (72) to the simple quadratic equation

$$\lambda^2 - (\lambda_0 + \lambda_c - kQ\varepsilon\tau)\lambda + \lambda_0 \lambda_c = 0. \quad (74)$$

The solutions of this equation are

$$\lambda_{1,2} = \frac{\lambda_0 + \lambda_c - kQ\varepsilon\tau}{2} \pm \sqrt{\frac{(\lambda_0 + \lambda_c - kQ\varepsilon\tau)^2}{4} - \lambda_0 \lambda_c}. \quad (75)$$

In Fig. 1(a), we compare the leading Floquet exponents of the controlled system determined by three different methods, namely, (i) by solving the quasipolynomial Eq. (72), (ii) using the solutions (75) of the quadratic equation (74), and (iii) by solving the exact system of variational Eqs. (47). Equation (72) has been solved by the Newton-Raphson algorithm. The numerical analysis of the variational Eqs. (47) has been performed by the algorithm described in Ref. [25]. All three above results are in good quantitative agreement, as viewed in Fig. 1(a). Thus the leading Floquet exponents are reliably predicted by the simple analytical expression (75).

The mechanism of stabilization is evident from Fig. 1(b). For $k=0$, two real positive solutions of Eq. (74), $\lambda = \lambda_0$ and $\lambda = \lambda_c$, describe an unstable eigenvalue of the free system and the free controller, respectively. With increasing k , the eigenvalues approach each other on the real axis, then collide and pass to the complex plane. For $k=k_0$, where

$$k_0 = (\lambda_0 + \lambda_c)/Q\varepsilon\tau, \quad (76)$$

they cross the imaginary axis and move symmetrically into the left half plane, i.e., both the system and the controller become stable. An optimal value of the control gain is

$$k_{op} = k_0 + 2\sqrt{\lambda_0 \lambda_c} Q\varepsilon\tau, \quad (77)$$

since it provides the fastest convergence to the stabilized limit cycle with the characteristic rate $\lambda_{min} = -\sqrt{\lambda_0 \lambda_c}$.

E. Numerical demonstrations

To verify the validity of the linear theory we have numerically investigated the original system of nonlinear differential-difference Eqs. (45). For the set of parameters $\varepsilon=0.1$ ($r \approx 24.143\,890\,65$), $\lambda_c=0.2$, $k=9.25$, τ

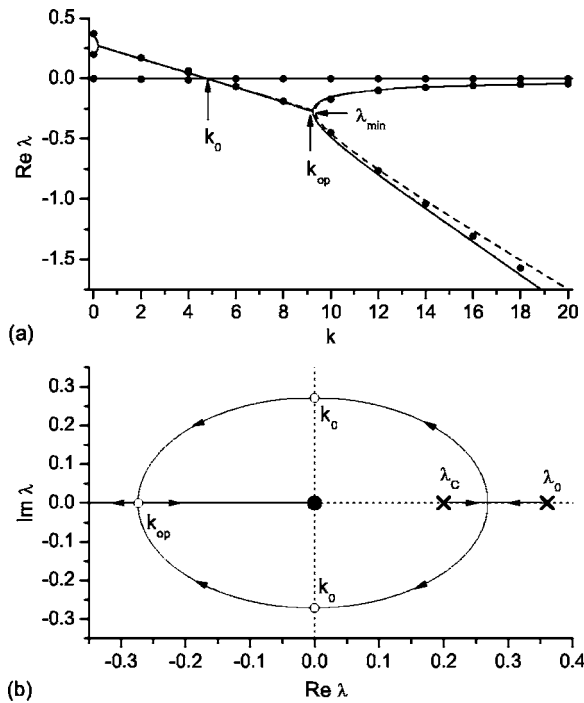


FIG. 1. (a) Real parts of leading Floquet exponents of the controlled limit cycle as functions of the control gain for $\varepsilon=0.1$, $\lambda_c=0.2$. Dashed and solid lines show the solutions of the characteristic Eqs. (74) and (72), respectively. Dots correspond to the values of Floquet exponents obtained from the exact variational Eqs. (47). (b) Root loci of Eq. (72) as k varies from 0 to ∞ for the same parameter values as in (a). Crosses and black dot denote the location of the roots for $k=0$ and ∞ , respectively.

$\approx 0.673\,983\,28$, the results are presented in Fig. 2. In numerical simulations, the controller is switched on only when the system is close to the desired periodic orbit and switched off when it is far away from the orbit. Specifically, we proceed in the following way. For large values of the quantity $|y-y_\tau| > Y_{max}=1.2$, we turn off the controller, i.e., we take $k=0$ and eliminate the term $\varepsilon w(y-y_\tau)$ in Eq. (45b). The controller variable is dropped to zero, $w=0$, at every moment of the turning off.

Without control ($t < \tau$), the Lorenz system demonstrates a chaotic behavior on the strange attractor. For $t > \tau$, the control algorithm starts to act and after a transient process the controlled system approaches a previously unstable limit cycle, and the feedback perturbation vanishes.

V. CONCLUSIONS

We have developed an analytical theory of the unstable delayed feedback controller recently proposed in the letter [20] for stabilization of unstable periodic orbits without torsion. The theory is applicable for any dynamical system close to a subcritical Hopf bifurcation. The periodic orbits arising at this bifurcation satisfy the odd number limitation

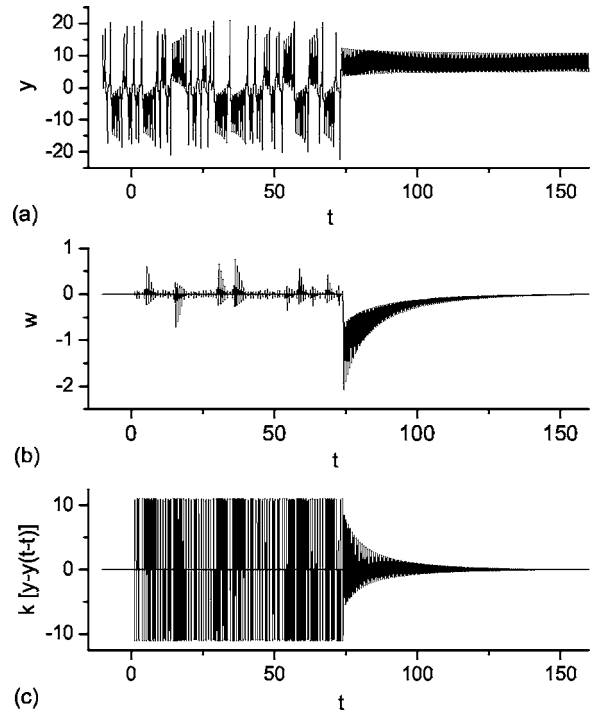


FIG. 2. Dynamics of (a) variable y , (b) controller variable w , and (c) delayed feedback perturbation $k[y-y(t-\tau)]$. The initial conditions are $x(-15\tau)=8.109\,559\,459$, $y(-15\tau)=13.037\,199\,46$, $z(-15\tau)=14.274\,650\,65$, $w(-15\tau)=0$, $y(t)=0$ for $-15\tau < t \leq -14\tau$. The control is initiated at $t=\tau$. The values of the parameters are $\varepsilon=0.1$, $\lambda_c=0.2$, $\tau=0.673\,983\,28$, $k=0$ for $-15\tau \leq t < \tau$ and $k=9.25$ for $t \geq \tau$. For $|y-y(t-\tau)| > Y_{max}=1.2$, the controller is off (see Sec. IV E for details). The black regions are densely filled by oscillations.

and could not be stabilized by the conventional delay technique. Our analytical approach is demonstrated with the Lorenz system.

To compare our approach to that presented in Ref. [18] we note that the simple second-order dynamical system under delayed feedback control considered in Ref. [18] has been treated analytically by the averaging method. Here, in the case of a more complex third-order dynamical system, we had to utilize additional tools of nonlinear dynamics, namely, the center manifold theory and the near identity transformation. It should be recognized that instead of the near identity transformation the multiscaling approach can be used. Both approaches lead to the same results, but the first approach is simpler. By using these tools we managed to derive analytically the characteristic equation for the Floquet exponents of the Lorenz system under delayed feedback control. Solving this equation we have determined simple analytical expressions for the leading Floquet exponents as well as for the threshold of stability and the optimal value of the control gain. Although the analytical approach has been demonstrated for a specific third-order system its extension to systems with the phase space dimension higher than 3 is straightforward.

We emphasize that the theory of the delayed feedback

control is very complicated. Therefore any analytical results even though they are elaborated for a particular class of dynamic systems represent a valuable contribution to the theory of delayed feedback control. The analytical results obtained

in this paper give a better insight into the mechanism of the delayed feedback control of unstable periodic orbits without torsion and are important for optimizing the control technique.

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