Time-Delayed Feedback Control Method and Unstable Controllers

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Abstract

Time delayed-feedback control is an efficient method for stabilizing unstable periodic orbits of chaotic systems. The method is based on applying feedback proportional to the deviation of the current state of the system from its state one period in the past so that the control signal vanishes when the stabilization of the desired orbit is attained. A brief review of experimental implementations, applications for theoretical models, most important modifications as well as recent advancements in the theory of the method is presented. An idea of using unstable degrees of freedom in a feedback loop to avoid a well known topological limitation of the method is described in details. This idea is extended for the problem of adaptive stabilization of unknown steady states of dynamical systems.

1 Introduction

An idea of controlling chaos has been first formulated by Ott, Grebogi, and Yorke [1] and attracted great interest among physicists over the past decade. Wy chaotic systems are interesting subjects for control theory and applications? The major key ingredient for the control of chaos is the observation that a chaotic set, on which the trajectory of the chaotic process lives, has embedded within it a large number of unstable periodic orbits (UPOs). In addition, because of ergodicity, the trajectory visits or accesses the neighborhood of each one of these periodic orbits. Some of these periodic orbits may correspond to a desired system's performance according to some criterion. The second ingredient is the realization that chaos, while signifying sensitive dependence on small changes to the current state and henceforth rendering unpredictable the system state in the long time, also implies that the system's behavior can be altered by using small perturbations. Then the accessibility of the chaotic system to many different period orbits combined with its sensitivity to small perturbations allows for the control and manipulation of the chaotic process. These ideas stimulated a development of rich variety of new chaos control techniques (see Refs. [2]–[4] for review), among which the delayed feedback control (DFC) method [5] has gained widespread acceptance.

The DFC method is based on applying feedback proportional to the deviation of the current state of the system from its state one period in the past so that the control signal vanishes when the stabilization of the desired orbit is attained. Alternatively the DFC method is referred to as a method of time-delay autosynchronization, since the stabilization of the desired orbit manifests itself as a synchronization of the current state of the system with its delayed state. The DFC has the advantage of not requiring prior knowledge of anything except the period of the desired orbit. It is particularly convenient for fast dynamical systems since does not require the real-time computer processing. Experimental implementations, applications for theoretical models, most important modifications as as well as recent advancements in the theory of the DFC method are briefly listed below.

Experimental implementations.— The time-delayed feedback control has been successfully used in quite diverse experimental contexts including electronic chaos oscillators [6]–[9], mechanical pendulums [10]–[11], lasers [12]–[14], a gas discharge system [15]–[17], a current-driven ion acoustic instability [18], a chaotic Taylor-Couette flow [19], chemical systems [20]–[21], high-power ferromagnetic resonance [22], helicopter rotor blades [23], and a cardiac system [24].

Applications for theoretical models.— The DFC method has been verified for a large number of theoretical models from different fields. Simmendinger and Hess [25] proposed an all-optical scheme based on the DFC for controlling delay-induced chaotic behavior of high-speed semiconductor lasers. The problem of stabilizing semiconductor laser arrays has been considered as well [26]–[27]. Rappel, Fenton, and Karma [28] used the DFC for stabilization of spiral waves in an excitable media as a model of cardiac tissue in order to prevent the spiral wave breakup. Konishi, Kokame, and Hirata [29]–[30] applied the DFC in a model of a car-following traffic. Batlle, Fossas, and Olivar [31] implemented the DFC in a model of buck converter. Bleich and Socolar [32] showed that the DFC can stabilize regular behavior in a paced, excitable oscillator described by Fitzhugh-Nagumo equations. Holyst, Zebrowska, and Urbanowicz [33]–[34] used the DFC to control chaos in economical model. Tsui and Jones investigated the problem of chaotic satellite attitude control [35] and constructed a feedforward neural network with the DFC to demonstrate a retrieval behavior that is analogous to the act of recognition [36]. The problem of controlling chaotic solitons by a time-delayed feedback mechanism has been considered by Fronczak and Holyst [37]. Mensour and Longtin [38] proposed to use the DFC in order to store information in delay-differential equations. Galvanetto [39] demonstrated the delayed feedback control of chaotic systems with dry friction. Lastly, Mitsubori and Aihara [40] proposed rather exotic application of the DFC, namely, the control of chaotic roll motion of a flooded ship in waves.

Modifications.— A reach variety of modifications of the DFC have been suggested in order to improve its performance. Adaptive versions of the DFC with automatic adjustment of delay time [41]–[43] and control gain [44]–[45] have been considered. Basso *et al.* [46]–[47] showed that for a Lur'e system (system represented as feedback connection of a linear dynamical part and a static nonlinearity) the DFC can be optimized by introducing into a feedback loop a linear filter with an appropriate transfer function. For spatially extended systems, various modifications based on spatially filtered signals have been considered [48]–[50]. The wave character of dynamics in some systems allows a simplification of the DFC algorithm by replacing the delay line with the spatially distributed detectors. Mausbach *et al.* [17] reported such a simplification for a ionization wave experiment in a conventional cold cathode glow discharge tube. Due to dispersion relations the delay in time is equivalent to the spatial displacement and the control signal can be constructed without use of the delay line. Socolar, Sukow, and Gauthier [51] improved an original DFC scheme by using an information from many previous states of the system. This extended DFC (EDFC) scheme achieves stabilization of UPOs with a greater degree of instability [52]–[53]. The EDFC presumably is the most important modification of the DFC and it will be discussed at greater length in this paper.

Advancements in the theory.—The theory of the DFC is rather intricate since it involves nonlinear delay-differential equations. If the equations governing the system dynamics are known the success of the DFC method can be predicted by a linear stability analysis of the desired orbit. Several numerical methods for the linear stability analysis of time-delayed feedback systems have been developed. The main difficulty of this analysis is related to the fact that periodic solutions of such systems have an infinite number of Floquet exponents (FEs), though only several FEs with the largest real parts are relevant for stability properties. Most straightforward method for evaluating several largest FEs is described in Ref. [52]. It adapts the usual procedure of estimating the Lyapunov exponents of strange attractors. This method requires a numerical integration of the variational system of delay-differential equations. Bleich and Socolar [53] devised an elegant method to obtain the stability domain of the system under EDFC in which the delay terms in variational equations are eliminated due to the Floquet theorem and the explicit integration of time-delay equations is avoided. Unfortunately, this method does not define the values of the FEs inside the stability domain and is unsuitable for optimization problems. An approximate analytical method for estimating the FEs of time-delayed feedback systems has been developed in Refs. [54]–[56]. Here as well as in Ref. [53] the delay terms in variational equations are eliminated and the Floquet problem is reduced to the system of ordinary differential equations. However, the FEs of the reduced system depend on a parameter that is a function of the unknown FEs itself. In Refs. [54]–[56] the problem is solved on the assumption that the FE of the reduced system depends linearly on the parameter. This method gives a better insight into mechanism of the DFC and leads to reasonable qualitative results. Some recent theoretical results on the DFC method are presented in Ref. [57]. Here it is shown that the main stability properties of the system controlled by time-delayed feedback can be simply derived from a leading FE defining the system behavior under proportional feedback control. As a result the optimal parameters of the delayed feedback controller can be evaluated without an explicit integration of delay-differential equations.

Although linear stability analysis of the delayed feedback systems is difficult, some general analytical results have been obtained [54],[58]–[60]. It has been shown that the DFC can stabilize only a certain class of periodic orbits characterized by a finite torsion.

More precisely, the limitation is that any UPOs with an odd number of real Floquet multipliers (FMs) greater than unity (or with an odd number of real positive FEs) can never be stabilized by the DFC. This statement was first proved by Ushio [58] for discrete time systems. Just *et al.* [54] and Nakajima [59] proved the same limitation for the continuous time DFC, and then this proof was extended for a wider class of delayed feedback schemes, including the EDFC [60]. Hence it seems hard to overcome this inherent limitation. Two efforts based on an oscillating feedback [61]–[62] and a half-period delay [63] have been taken to obviate this drawback. In both cases the mechanism of stabilization is rather unclear. Besides, the method of Ref. [63] is valid only for a special case of symmetric orbits. The limitation has been recently eliminated in a new modification of the DFC that does not utilize the symmetry of UPOs [64]. The key idea is to introduce into a feedback loop an additional unstable degree of freedom that changes the total number of unstable torsion-free modes to an even number. Then the idea of using unstable degrees of freedom in a feedback loop was drown on to construct a simple adaptive controller for stabilizing unknown steady states of dynamical systems [65].

Some recent theoretical results on the unstable controller are presented in more details in the rest of the paper. In Section 2 the problem of stabilizing torsion-free periodic orbits is considered. We start with a simple discrete time model and show that an unstable degree of freedom introduced into a feedback loop can overcome the limitation of the DFC method. Then we propose a generalized modification of the DFC for torsion-free UPOs and demonstrate its efficiency for the Lorenz system. Section 3 is devoted to the problem of adaptive stabilization of unknown steady states of dynamical systems. We propose an adaptive controller described by ordinary differential equations and prove that the steady state can never be stabilized if the system and controller in sum have an odd number of real positive eigenvalues. We show that the adaptive stabilization of saddle-type steady states requires the presence of an unstable degree of freedom in a feedback loop. The paper is finished with conclusions presented in Section 4.

2 Stabilizing torsion-free periodic orbits

Most investigations on the DFC method are restricted to the consideration of unstable periodic orbits erasing from a flip bifurcation. The leading Floquet multiplier of such orbits is real and negative (or corresponding FE lies on the boundary of the "Brillouin zone", $\mathrm{Im}\Lambda = \pi/T$). Such a consideration is motivated by the fact that the usual DFC and EDFC methods work only for the orbits with a finite torsion, when the leading FE obeys $\mathrm{Im}\Lambda \neq 0$. Unsuitability of the DFC technique to stabilize torsion-free orbits ($\mathrm{Im}\Lambda = 0$) has been over several years considered as a main limitation of the method [54],[58]–[60]. More precisely, the limitation is that any UPOs with an odd number of real Floquet multipliers greater than unity can never be stabilized by the DFC. This limitation can be explained by bifurcation theory as follows. When an UPO with an odd number of real FMs greater than unity is stabilized, one of such multipliers must cross the unite circle on the real axes in the complex plane. Such a situation correspond to a tangent bifurcation, which is accompanied with a coalescence of T-periodic orbits. However, this contradicts the fact that DFC perturbation does not change the location of T-periodic orbits when the feedback gain varies, because the feedback term vanishes for T-periodic orbits.

Here we describe an unstable delayed feedback controller that can overcome the limitation. The idea is to artificially enlarge a set of real multipliers greater than unity to an even number by introducing into a feedback loop an unstable degree of freedom.

2.1 Simple example: EDFC for $R > 1$

First we illustrate the idea for a simple unstable discrete time system $y_{n+1} = \mu_s y_n$, $\mu_s > 1$ controlled by the EDFC:

$$y_{n+1} = \mu_s y_n - KF_n, \quad (1)$$

$$F_n = y_n - y_{n-1} + RF_{n-1}. \quad (2)$$

The free system $y_{n+1} = \mu_s y_n$ has an unstable fixed point $y^\star = 0$ with the only real eigenvalue $\mu_s > 1$ and, in accordance with the above limitation, can not be stabilized by the EDFC for any values of the feedback gain K. This is so indeed if the EDFC is stable, i.e., if the parameter R in Eq. (2) satisfies the inequality $|R| < 1$. Only this case has been considered in the literature. However, it is easy to show that the unstable controller with the parameter $R > 1$ can stabilize this system. Using the ansatz $y_n, F_n \propto \mu^n$ one obtains the characteristic equation

$$(\mu - \mu_s)(\mu - R) + K(\mu - 1) = 0 \quad (3)$$

defining the eigenvalues μ of the closed loop system (1,2). The system is stable if both roots $\mu = \mu_{1,2}$ of Eq. (3) are inside the unit circle of the μ complex plain, $|\mu_{1,2}| < 1$. Figure 1 (a) shows the characteristic root-locus diagram for $R > 1$, as the parameter K varies from 0 to ∞. For $K = 0$, there are two real eigenvalues greater than unity, $\mu_1 = \mu_s$ and

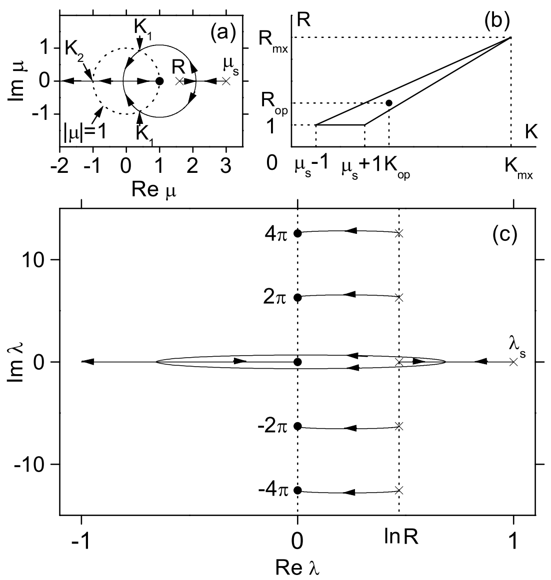

Figure 1: Performance of (a,b) discrete and (c) continuous EDFC for $R > 1$. (a) Root loci of Eq. (3) at $\mu_s = 3$, $R = 1.6$ as K varies from 0 to ∞. (b) Stability domain of Eqs. (1,2) in the (K, R) plane; $K_{mx} = (\mu_s+1)^2/(\mu_s-1)$, $R_{mx} = (\mu_s+3)/(\mu_s-1)$. (c) Root loci of Eq. (6) at $\lambda_s = 1$, $R = 1.6$. The crosses and circles denote the location of roots at $K = 0$ and $K \to \infty$, respectively.

$\mu_2 = R$, which correspond to two independent subsystems (1) and (2), respectively; this means that both the controlled system and controller are unstable. With the increase of K, the eigenvalues approach each other on the real axes, then collide and pass to the complex plain. At $K = K_1 \equiv \mu_s R - 1$ they cross symmetrically the unite circle $|\mu| = 1$. Then both eigenvalues move inside this circle, collide again on the real axes and one of them leaves the circle at $K = K_2 \equiv (\mu_s + 1)(R + 1)/2$. In the interval $K_1 < K < K_2$, the closed loop system (1,2) is stable. By a proper choice of the parameters R and K one can stabilize the fixed point with an arbitrarily large eigenvalue μ_s. The corresponding stability domain is shown in Fig. 1 (b). For a given value μ_s, there is an optimal choice of the parameters $R = R_{op} \equiv \mu_s/(\mu_s - 1)$, $K = K_{op} \equiv \mu_s R_{op}$ leading to zero eigenvalues, $\mu_1 = \mu_2 = 0$, such that the system approaches the fixed point in finite time.

It seems attractive to apply the EDFC with the parameter $R > 1$ for continuous time systems. Unfortunately, this idea fails. As an illustration, let us consider a continuous time version of Eqs. (1,2)

$$\dot{y}(t) = \lambda_s y(t) - KF(t), \quad (4)$$

$$F(t) = y(t) - y(t-\tau) + RF(t-\tau), \quad (5)$$

where $\lambda_s > 0$ is the characteristic exponent of the free system $\dot{y} = \lambda_s y$ and τ is the delay time. By a suitable rescaling one can eliminate one of the parameters in Eqs. (4,5). Thus, without a loss of generality we can take $\tau = 1$. Equations (4,5) can be solved by the Laplace transform or simply by the substitution $y(t), F(t) \propto e^{\lambda t}$, that yields the characteristic equation:

$$1 + K\frac{1-\exp(-\lambda)}{1-R\exp(-\lambda)}\frac{1}{\lambda - \lambda_s} = 0. \quad (6)$$

In terms of the control theory, Eq. (6) defines the poles of the closed loop transfer function. The first and second fractions in Eq. (6) correspond to the EDFC and plant transfer functions, respectively. The closed loop system (4,5) is stable if all the roots of Eq. (6) are in the left half-plane, $\mathrm{Re}\lambda < 0$. The characteristic root-locus diagram for $R > 1$ is shown in Fig. 1 (c). When K varies from 0 to ∞, the EDFC roots move in the right half-plane from locations $\lambda = \ln R + 2\pi i n$ to $\lambda = 2\pi i n$ for $n = \pm 1, \pm 2 \ldots$. Thus, the continuous time EDFC with the parameter $R > 1$ has an infinite number of unstable degrees of freedom and many of them remain unstable in the closed loop system for any K.

2.2 Usual EDFC supplemented by an unstable degree of freedom

Hereafter, we use the usual EDFC at $0 \le R < 1$, however introduce an additional unstable degree of freedom into a feedback loop. More specifically, for a dynamical system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, p)$ with a measurable scalar variable $y(t) = g(\boldsymbol{x}(t))$ and an UPO of period τ at $p = 0$, we propose to adjust an available system parameter p by a feedback signal $p(t) = KF_u(t)$ of the following form:

$$F_u(t) = F(t) + w(t), \quad (7)$$

$$\dot{w}(t) = \lambda_c^0 w(t) + (\lambda_c^0 - \lambda_c^\infty)F(t), \quad (8)$$

$$F(t) = y(t) - (1-R)\sum_{k=1}^{\infty} R^{k-1} y(t-k\tau), \quad (9)$$

where $F(t)$ is the usual EDFC described by Eq. (5) or equivalently by Eq. (9). Equation (8) defines an additional unstable degree of freedom with parameters $\lambda_c^0 > 0$ and $\lambda_c^\infty < 0$. We emphasize that whenever the stabilization is successful the variables $F(t)$ and $w(t)$ vanish, and thus vanishes the feedback force $F_u(t)$. We refer to the feedback law (7–9) as an unstable EDFC (UEDFC).

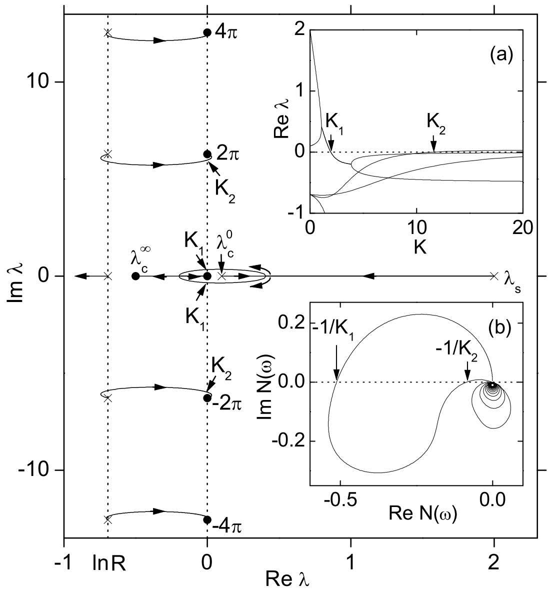

Figure 2: Root loci of Eq. (11) at $\lambda_s = 2$, $\lambda_c^0 = 0.1$, $\lambda_c^\infty = -0.5$, $R = 0.5$. The insets (a) and (b) show Reλ vs. K and the Nyquist plot, respectively. The boundaries of the stability domain are $K_1 \approx 1.95$ and $K_2 \approx 11.6$.

To get an insight into how the UEDFC works let us consider again the problem of stabilizing the fix point

$$\dot{y} = \lambda_s y - KF_u(t), \tag{10}$$

where $F_u(t)$ is defined by Eqs. (7–9) and $\lambda_s > 0$. Here as well as in a previous example we can take $\tau = 1$ without a loss of generality. Now the characteristic equation reads:

$$1 + KQ(\lambda) = 0, \tag{11}$$

$$Q(\lambda) \equiv \frac{\lambda - \lambda_c^\infty}{\lambda - \lambda_c^0} \frac{1 - \exp(-\lambda)}{1 - R\exp(-\lambda)} \frac{1}{\lambda - \lambda_s}. \tag{12}$$

The first fraction in Eq. (12) corresponds to the transfer function of an additional unstable degree of freedom. Root loci of Eq. (11) is shown in Fig. 2. The poles and zeros of Q-function define the value of roots at $K = 0$ and $K \to \infty$, respectively. Now at $K = 0$, the EDFC roots $\lambda = lnR + 2\pi in$, $n = 0, \pm1, \ldots$ are in the left half-plane. The only root λ_c^0 associated with an additional unstable degree of freedom is in the right half-plane. That root and the root λ_s of the fix point collide on the real axes, pass to the complex plane and at $K = K_1$ cross into the left half-plane. For $K_1 < K < K_2$, all roots of Eq. (11) satisfy the inequality Re$\lambda < 0$, and the closed loop system (7–10) is stable. The stability is destroyed at $K = K_2$ when the EDFC roots $\lambda = \ln R \pm 2\pi i$ in the second "Brillouin zone" cross into Re$\lambda > 0$. The dependence of the five largest Reλ on K is shown in the inset (a) of Fig. 2. The inset (b) shows the Nyquist plot, i.e., a parametric plot Re$N(\omega)$ versus Im$N(\omega)$ for $\omega \in [0, \infty]$, where $N(\omega) \equiv Q(i\omega)$. The Nyquist plot provides the simplest way of determining the stability domain; it crosses the real axes at Re$N = -1/K_1$ and Re$N = -1/K_2$.

As a more involved example let us consider the Lorenz system under the UEDFC:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} -\sigma x + \sigma y \\ rx - y - xz \\ xy - bz \end{pmatrix} - KF_u(t) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}. \tag{13}$$

We assume that the output variable is y and the feedback force $F_u(t)$ [Eqs. (7–9)] perturbs only the second equation of the Lorenz system. Denote the variables of the Lorenz system by $\boldsymbol{\rho} = (x, y, z)$ and those extended with the controller variable w by $\boldsymbol{\xi} = (\boldsymbol{\rho}, w)^T$. For the parameters $\sigma = 10$, $r = 28$, and $b = 8/3$, the free ($K = 0$) Lorenz system has a period-one UPO, $\boldsymbol{\rho}_0(t) \equiv (x_0, y_0, z_0) = \boldsymbol{\rho}_0(t+\tau)$, with the period $\tau \approx 1.5586$ and all real FMs: $\mu_1 \approx 4.714$, $\mu_2 = 1$ and $\mu_3 \approx 1.19 \times 10^{-10}$. This orbit can not be stabilized by usual DFC or EDFC, since only one FM is greater than unity. The ability of the UEDFC to stabilize this orbit can be verified by a linear analysis of Eqs. (13) and (7–9). Small deviations $\delta\boldsymbol{\xi} = \boldsymbol{\xi} - \boldsymbol{\xi}_0$ from the periodic solution $\boldsymbol{\xi}_0(t) \equiv (\boldsymbol{\rho}_0, 0)^T = \boldsymbol{\xi}_0(t + \tau)$ may be decomposed into eigenfunctions according to the Floquet theory, $\delta\boldsymbol{\xi} = e^{\lambda t}\boldsymbol{u}$, $\boldsymbol{u}(t) = \boldsymbol{u}(t + \tau)$, where λ is the Floquet exponent. The Floquet decomposition yields linear periodically time dependent equations $\delta\dot{\boldsymbol{\xi}} = A\delta\boldsymbol{\xi}$ with the boundary condition $\delta\boldsymbol{\xi}(\tau) = e^{\lambda\tau}\delta\boldsymbol{\xi}(0)$, where

$$A = \begin{pmatrix} -\sigma & \sigma & 0 & 0 \\ r - z_0(t) & -(1 + KH) & -x_0(t) & -K \\ y_0(t) & x_0(t) & -b & 0 \\ 0 & (\lambda_c^0 - \lambda_c^\infty)H & 0 & \lambda_c^0 \end{pmatrix}. \tag{14}$$

Due to equality $\delta y(t - k\tau) = e^{-k\lambda\tau}\delta y(t)$, the delay terms in Eq. (9) are eliminated, and Eq. (9) is transformed to $\delta F(t) = H\delta y(t)$, where

$$H = H(\lambda) = (1 - exp(-\lambda\tau))/(1 - R\exp(-\lambda\tau)) \tag{15}$$

is the transfer function of the EDFC. The price for this simplification is that the Jacobian A, defining the exponents λ, depends on λ itself. The eigenvalue problem may be solved with an evolution matrix Φ_t that satisfies

$$\dot{\Phi}_t = A\Phi_t, \quad \Phi_0 = I. \tag{16}$$

The eigenvalues of Φ_τ define the desired exponents:

$$\det[\Phi_\tau(H) - e^{\lambda\tau}I] = 0. \tag{17}$$

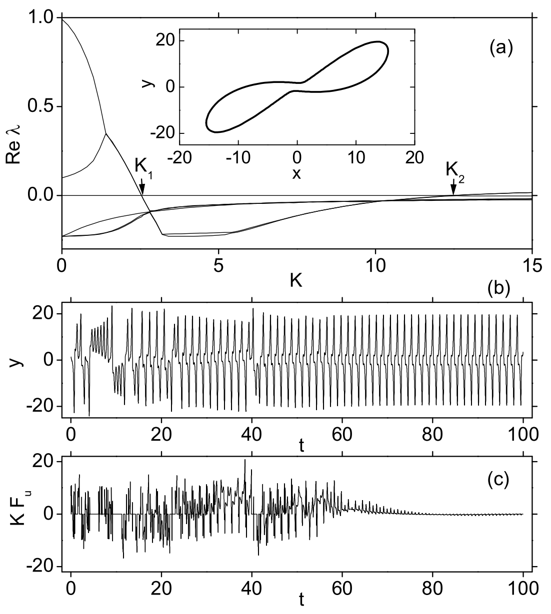

Figure 3: Stabilizing an UPO of the Lorenz system. (a) Six largest Reλ vs. K. The boundaries of the stability domain are $K_1 \approx 2.54$ and $K_2 \approx 12.3$. The inset shows the (x, y) projection of the UPO. (b) and (c) shows the dynamics of $y(t)$ and $F_u(t)$ obtained from Eqs. (13,7–9). The parameters are: $\lambda_c^0 = 0.1$, $\lambda_c^\infty = -2$, $R = 0.7$, $K = 3.5$, $\varepsilon = 3$, $\lambda_r = 10$.

We emphasize the dependence Φ_τ on H conditioned by the dependence of A on H. Thus by solving Eqs. (15–17), one can define the Floquet exponents λ (or multipliers $\mu = e^{\lambda\tau}$) of the Lorenz system under the UEDFC. Figure 3 (a) shows the dependence of the six largest Reλ on K. There is an interval $K_1 < K < K_2$, where the real parts of all exponents are negative. Basically, Fig. 3 (a) shows the results similar to those presented in Fig. 2 (a). The unstable exponent λ_1 of an UPO and the unstable eigenvalue λ_c^0 of the controller collide on the real axes and pass into the complex plane providing an UPO with a finite torsion. Then this pair of complex conjugate exponents cross into domain Re$\lambda < 0$, just as they do in the simple model of Eq. (10).

Direct integration of the nonlinear Eqs. (13, 7–9) confirms the results of linear analysis. Figures 3 (b,c) show a successful stabilization of the desired UPO with an asymptotically vanishing perturbation. In this analysis, we used a restricted perturbation similar as we did in Ref. [5]. For $|F(t)| < \varepsilon$, the control force $F_u(t)$ is calculated from Eqs. (7–9), however for $|F(t)| > \varepsilon$, the control is switched off, $F_u(t) = 0$, and the unstable variable w is dropped off by replacing Eq. (8) with the relaxation equation $\dot{w} = -\lambda_r w$, $\lambda_r > 0$.

To verify the influence of fluctuations a small white noise with the spectral density $S(\omega) = a$ has been added to the r.h.s. of Eqs. (8,13). At every step of integration the variables x, y, z, and w were shifted by an amount $\sqrt{12ha}\xi_i$, where ξ_i are the random numbers uniformly distributed in the interval $[-0.5, 0.5]$ and h is the stepsize of integration. The control method works when the noise is increased up to $a \approx 0.02$. The variance of perturbation increases proportionally to the noise amplitude, $\langle F_u^2(t)\rangle = ka$, $k \approx 17$. For a large noise $a > 0.02$, the system intermittently loses the desired orbit.

3 Stabilizing and tracking unknown steady states

Although the field of controlling chaos deals mainly with the stabilization of unstable periodic orbits, the problem of stabilizing unstable steady states of dynamical systems is of great importance for various technical applications. Stabilization of a fixed point by usual methods of classical control theory requires a knowledge of its location in the phase space. However, for many complex systems (e.g., chemical or biological) the location of the fixed points, as well as exact model equations, are unknown. In this case adaptive control techniques capable of automatically locating the unknown steady state are preferable. An adaptive stabilization of a fixed point can be attained with the time-delayed feedback method [5],[52],[66]. However, the use of time-delayed signals in this problem is not necessary and thus the difficulties related to an infinite dimensional phase space due to delay can be avoided. A simpler adaptive controller for stabilizing unknown steady states can be designed on a basis of ordinary differential equations (ODEs). The simplest example of such a controller utilizes a conventional low pass filter described by one ODE. The filtered dc output signal of the system estimates the location of the fixed point, so that the difference between the actual and filtered output signals can be used as a control signal. An efficiency of such a simple controller has been demonstrated for different experimental systems [66]–[68]. Further examples involve methods which do not require knowledge of the position of the steady state but result in a nonzero control signal [69].

In this section we describe a generalized adaptive controller characterized by a system of ODEs and prove that it has a topological limitation concerning an odd number of real positive eigenvalues of the steady state [65]. We show that the limitation can

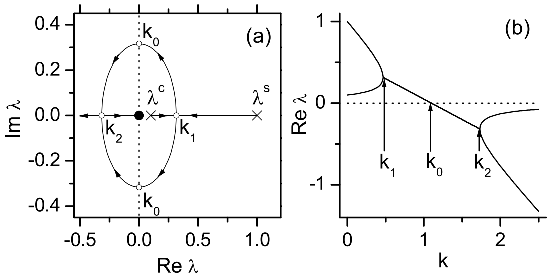

Figure 4: Stabilizing an unstable fixed point with an unstable controller in a simple model of Eqs. (18) for $\lambda^s = 1$ and $\lambda^c = 0.1$. (a) Root loci of the characteristic equation as k varies from 0 to ∞. The crosses and solid dot denote the location of roots at $k = 0$ and $k \to \infty$, respectively. (b) Reλ vs. k. $k_0 = \lambda^s + \lambda^c$, $k_{1,2} = \lambda^s + \lambda^c \mp 2\sqrt{\lambda^s \lambda^c}$.

be overcome by implementing an unstable degree of freedom into a feedback loop. The feedback produces a robust method of stabilizing a priori unknown unstable steady states, saddles, foci, and nodes.

3.1 Simple example

An adaptive controller based on the conventional low-pass filter, successfully used in several experiments [66], is not universal. This can be illustrated with a simple model:

$$\dot{x} = \lambda^s(x - x^\star) + k(w - x), \quad \dot{w} = \lambda^c(w - x). \quad (18)$$

Here x is a scalar variable of an unstable one-dimensional dynamical system $\dot{x} = \lambda^s(x - x^\star)$, $\lambda^s > 0$ that we intend to stabilize. We imagine that the location of the fixed point $x^\star$ is unknown and use a feedback signal $k(w - x)$ for stabilization. The equation $\dot{w} = \lambda^c(w - x)$ for $\lambda^c < 0$ represents a conventional low-pass filter (rc circuit) with a time constant $\tau = -1/\lambda^c$. The fixed point of the closed loop system in the whole phase space of variables (x, w) is $(x^\star, x^\star)$ so that its projection on the x axes corresponds to the fixed point of the free system for any control gain k. If for some values of k the closed loop system is stable, the controller variable w converges to the steady state value $w^\star = x^\star$ and the feedback perturbation vanishes.

The closed loop system is stable if both eigenvalues of the characteristic equation $\lambda^2 - (\lambda^s + \lambda^c - k)\lambda + \lambda^s\lambda^c = 0$ are in the left half-plane Re$\lambda < 0$. The stability conditions are: $k > \lambda^s + \lambda^c$, $\lambda^s\lambda^c > 0$. We see immediately that the stabilization is not possible with a conventional low-pass filter since for any $\lambda^s > 0$, $\lambda^c < 0$, we have $\lambda^s\lambda^c < 0$ and the second stability criterion is not met. However, the stabilization can be attained via an unstable controller with a positive parameter λ^c. Electronically, such a controller can be devised as the RC circuit with a negative resistance. Figure 4 shows a mechanism of stabilization. For $k = 0$, the eigenvalues are λ^s and λ^c, which correspond to the free system and free controller, respectively. With the increase of k, they approach each other on the real axes, then collide at $k = k_1$ and pass to the complex plane. At $k = k_0$ they cross symmetrically into the left half-plane (Hopf bifurcation). At $k = k_2$ we have again a collision on the real axes and then one of the roots moves towards $-\infty$ and another approaches the origin. For $k > k_0$, the closed loop system is stable. An optimal value of the control gain is k_2 since it provides the fastest convergence to the fixed point.

3.2 Generalized adaptive controller

Now we consider the problem of adaptive stabilization of a steady state in general. Let

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{p}) \quad (19)$$

be the dynamical system with N-dimensional vector variable $\boldsymbol{x}$ and and L-dimensional vector parameter $\boldsymbol{p}$ available for an external adjustment. Assume that an n-dimensional vector variable $\boldsymbol{y}(t) = \boldsymbol{g}(\boldsymbol{x}(t))$ (a function of dynamical variables $\boldsymbol{x}(t)$) represents the system output. Suppose that at $\boldsymbol{p} = \boldsymbol{p}_0$ the system has an unstable fixed point $\boldsymbol{x}^\star$ that satisfies $\boldsymbol{f}(\boldsymbol{x}^\star, \boldsymbol{p}_0) = 0$. The location of the fixed point $\boldsymbol{x}^\star$ is unknown. To stabilize the fixed point we perturb the parameters by an adaptive feedback

$$\boldsymbol{p}(t) = \boldsymbol{p}_0 + kB[A\boldsymbol{w}(t) + C\boldsymbol{y}(t)] \quad (20)$$

where $\boldsymbol{w}$ is an M-dimensional dynamical variable of the controller that satisfies

$$\dot{\boldsymbol{w}}(t) = A\boldsymbol{w} + C\boldsymbol{y}. \quad (21)$$

Here A, B, and C are the matrices of dimensions $M \times M$, $M \times L$, and $n \times M$, respectively and k is a scalar parameter that defines the feedback gain. The feedback is constructed in such a way that it does not change the steady state solutions of the free system. For any k, the fixed point of the closed loop system in the whole phase space of variables $\{\boldsymbol{x}, \boldsymbol{w}\}$ is $\{\boldsymbol{x}^\star, \boldsymbol{w}^\star\}$, where $\boldsymbol{x}^\star$ is the fixed point of the free system and $\boldsymbol{w}^\star$ is the corresponding steady state value of the controller variable. The latter satisfies a system of linear equations $A\boldsymbol{w}^\star = -C\boldsymbol{g}(\boldsymbol{x}^\star)$ that has unique solution for any nonsingular matrix A. The feedback perturbation $kB\dot{\boldsymbol{w}}$ vanishes whenever the fixed point of the closed loop system is stabilized.

Small deviations $\delta\boldsymbol{x} = \boldsymbol{x} - \boldsymbol{x}^\star$ and $\delta\boldsymbol{w} = \boldsymbol{w} - \boldsymbol{w}^\star$ from the fixed point are described by variational equations

$$\delta\dot{\boldsymbol{x}} = J\delta\boldsymbol{x} + kPB\delta\dot{\boldsymbol{w}}, \quad \delta\dot{\boldsymbol{w}} = CG\delta\boldsymbol{x} + A\delta\boldsymbol{w}, \quad (22)$$

where $J = D_{\boldsymbol{x}} \boldsymbol{f}(\boldsymbol{x}^\star, \boldsymbol{p}_0)$, $P = D_{\boldsymbol{p}} \boldsymbol{f}(\boldsymbol{x}^\star, \boldsymbol{p}_0)$, and $G = D_{\boldsymbol{x}} \boldsymbol{g}(\boldsymbol{x}^\star)$. Here $D_{\boldsymbol{x}}$ and $D_{\boldsymbol{p}}$ denote the vector derivatives (Jacobian matrices) with respect to the variables $\boldsymbol{x}$ and parameters $\boldsymbol{p}$, respectively. The characteristic equation for the closed loop system reads:

$$\Delta_k(\lambda) \equiv \begin{vmatrix} I\lambda - J & -k\lambda PB \\ -CG & I\lambda - A \end{vmatrix} = 0. \qquad (23)$$

For $k = 0$ we have $\Delta_0(\lambda) = |I\lambda - J||I\lambda - A|$ and Eq. (23) splits into two independent equations $|I\lambda - J| = 0$ and $|I\lambda - A| = 0$ that define N eigenvalues of the free system $\lambda = \lambda_j^s$, $j = 1, \ldots, N$ and M eigenvalues of the free controller $\lambda = \lambda_j^c$, $j = 1, \ldots, M$, respectively. By assumption, at least one eigenvalue of the free system is in the right half-plane. The closed loop system is stabilized in an interval of the control gain k for which all eigenvalues of Eq. (23) are in the left half-plane $\mathrm{Re}\lambda < 0$.

The following theorem defines an important topological limitation of the above adaptive controller. It is similar to the Nakajima theorem [59] concerning the limitation of the time-delayed feedback controller.

Theorem.— Consider a fixed point $\boldsymbol{x}^\star$ of a dynamical system (19) characterized by Jacobian matrix J and an adaptive controller (21) with a nonsingular matrix A. If the total number of real positive eigenvalues of the matrices J and A is odd, then the closed loop system described by Eqs. (19)-(21) cannot be stabilized by any choice of matrices A, B, C and control gain k.

Proof.— The stability of the closed loop system is determined by the roots of $\Delta_k(\lambda)$. Writing Eq. (23) for $k = 0$ in the basis where martcies J and A are diagonal, we have

$$\Delta_0(\lambda) = \textstyle\prod_{j=1}^{N}(\lambda - \lambda_j^s) \prod_{m=1}^{M}(\lambda - \lambda_m^c). \qquad (24)$$

Here λ_j^s and λ_m^c are the eigenvalues of the matrices J and A, respectively. Now from Eq. (23), we also have $\Delta_k(0) = \Delta_0(0)$, so Eq. (24) implies

$$\Delta_k(0) = \textstyle\prod_{j=1}^{N}(-\lambda_j^s) \prod_{m=1}^{M}(-\lambda_m^c) \qquad (25)$$

for all k. Since the total number of eigenvalues λ_j^s and λ_m^c that are real and positive is odd and other eigenvalues are real and negative or come in complex conjugate pairs, $\Delta_k(0)$ must be real and negative. On the other hand, from the definition of $\Delta_k(\lambda)$ we see immediately that when $\lambda \to \infty$ then $\Delta_k(\lambda) \to \lambda^{N+M} > 0$ for all k. $\Delta_k(\lambda)$ is an $N + M$ order polynomial with real coefficients and is continuous for all λ. Since $\Delta_k(\lambda)$ is negative for $\lambda = 0$ and is positive for large λ, it follows that $\Delta_k(\lambda) = 0$ for some real positive λ. Thus the closed loop system always has at least one real positive eigenvalue and cannot be stabilized, Q.E.D.

This limitation can be explained by bifurcation theory, similar to Ref. [59]. If a fixed point with an odd total number of real positive eigenvalues is stabilized, one of such eigenvalues must cross into the left half-plane on the real axes accompanied with a coalescence of fixed points. However, this contradicts the fact that the feedback perturbation does not change locations of fixed points.

From this theorem it follows that any fixed point $\boldsymbol{x}^\star$ with an odd number of real positive eigenvalues cannot be stabilized with a stable controller. In other words, if the Jacobian J of a fixed point has an odd number of real positive eigenvalues then it can be stabilized only with an unstable controller whose matrix A has an odd number (at least one) of real positive eigenvalues.

3.3 Controlling an electrochemical oscillator

The use of an unstable degree of freedom in a feedback loop is now demonstrated with control in an electrodissolution process, the dissolution of nickel in sulfuric acid. The main features of this process can be qualitatively described with a model proposed by Haim *et al.* [70]. The dimensionless model together with the controller reads:

$$\dot{e} = i - (1 - \Theta)\left[\frac{C_h \exp(0.5e)}{1 + C_h \exp(e)} + a\exp(e)\right] \qquad (26)$$

$$\dot{\Theta} = \frac{1}{\Gamma}\left[\frac{\exp(0.5e)(1-\Theta)}{1 + C_h \exp(e)} - \frac{bC_h \exp(2e)\Theta}{C_h c + \exp(e)}\right] \qquad (27)$$

$$\dot{w} = \lambda^c(w - i) \qquad (28)$$

Here e is the dimensionless potential of the electrode and Θ is the surface coverage of NiO+NiOH. An observable is the current

$$i = (V_0 + \delta V - e)/R, \quad \delta V = k(i - w), \qquad (29)$$

where V_0 is the circuit potential and R is the series resistance of the cell. δV is the feedback perturbation applied to the circuit potential, k is the feedback gain. From Eqs. (29) it follows that $i = (V_0 - e - kw)/(R - k)$ and $\delta V = k(V_0 - e - wR)/(R - k)$. We see that the feedback perturbation is singular at $k = R$.

In a certain interval of the circuit potential V_0, a free ($\delta V = 0$) system has three coexisting fixed points: a stable node, a saddle, and an unstable focus [Fig. 5 (a)]. Depending on the initial conditions, the trajectories are attracted either to the stable node or to the stable limit cycle that surrounds an unstable focus. As is seen from Figs. 5 (b) and 5 (c) the coexisting saddle and the unstable focus can be stabilized

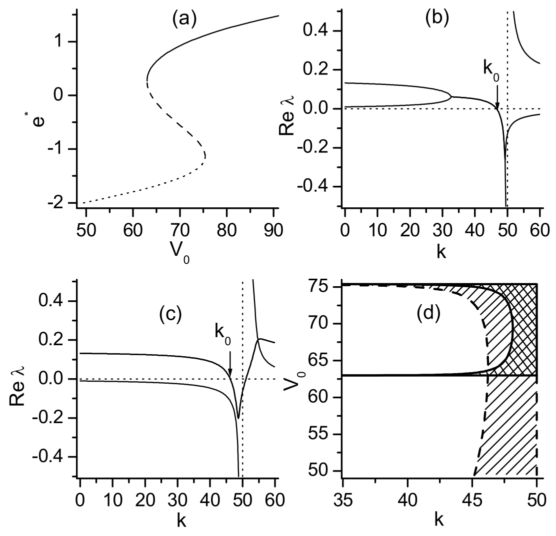

Figure 5: Results of analysis of the electrochemical model for $R = 50$, $C_h = 1600$, $a = 0.3$, $b = 6 \times 10^{-5}$, $c = 10^{-3}$, $\Gamma = 0.01$. (a) Steady solutions $e^\star$ vs. V_0 of the free ($\delta V = 0$) system. Solid, broken, and dotted curves correspond to a stable node, a saddle, and an unstable focus, respectively. (b) and (c) Eigenvalues of the closed loop system as functions of control gain k at $V_0 = 63.888$ for the saddle $(e^\star, \Theta^\star) = (0, 0.0166)$ controlled by an unstable controller ($\lambda^c = 0.01$) and for the unstable focus $(e^\star, \Theta^\star) = (-1.7074, 0.4521)$ controlled by a stable controller ($\lambda^c = -0.01$), respectively. (d) Stability domain in (k, V_0) plane for the saddle (crossed lines) at $\lambda^c = 0.01$ and for the focus (inclined lines) at $\lambda^c = -0.01$.

with the unstable ($\lambda^c > 0$) and stable ($\lambda^c < 0$) controller, respectively if the control gain is in the interval $k_0 < k < R = 50$. Figure 5 (d) shows the stability domains of these points in the (k, V_0) plane. If the value of the control gain is chosen close to $k = R$, the fixed points remain stable for all values of the potential V_0. This enables a tracking of the fixed points by fixing the control gain k and varying the potential V_0. In general a tracking algorithm requires a continuous updating of the target state and the control gain. Here described method finds the position of the steady states automatically. The method is robust enough in the examples investigated to operate without change in control gain. We also note that the stability of the saddle and focus points can be switched by a simple reversal of sign of the parameter λ^c.

Laboratory experiments for this system have been successfully carried out by I. Z. Kiss and J. L. Hudson [65]. They managed to stabilize and track both the unstable focus and the unstable saddle steady states. For the focus the usual rc circuit has been used, while the saddle point has been stabilized with the unstable controller. The robustness of the control algorithm allowed the stabilization of unstable steady states in a large parameter region. By mapping the stable and unstable phase objects the authors have visualized saddle-node and homoclinic bifurcations directly from experimental data.

4 Conclusions

The aim of this paper was to review experimental implementations, applications for theoretical models, and modifications of the time-delayed feedback control method and to present some recent theoretical ideas in this field.

In Section 2 we discussed the main limitation of the delayed feedback control method, which states that the method cannot stabilize torsion-free periodic orbits, ore more precisely, orbits with an odd number of real positive Flocke exponents. We have shown that this topological limitation can be eliminated by introduction into a feedback loop an unstable degree of freedom that changes the total number of unstable torsion-free modes to an even number. An efficiency of the modified scheme has been demonstrated for the Lorenz system. Note that the stability analysis of the torsion-free orbits controlled by unstable controller can be performed in a similar manner as described in Ref. [57]. This problem is currently under investigation and the results will be published elsewhere.

In Section 3 the idea of unstable controller has been used for the problem of stabilizing unknown steady states of dynamical systems. We have considered an adaptive controller described by a finite set of ordinary differential equations and proved that the steady state can never be stabilized if the system and controller in sum have an odd number of real positive eigenvalues. For two dimensional systems, this topological limitation states that only an unstable focus or node can be stabilized with a stable controller and stabilization of a saddle requires the presence of an unstable degree of freedom in a feedback loop. The use of the controller to stabilize and track saddle points (as well as unstable foci) has been demonstrated numerically with an electrochemical Ni dissolution system.

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