Control of Chaos via an Unstable Delayed Feedback Controller

K. Pyragas[1,2,*]
[1]Max-Planck-Institut für Physik komplexer Systeme, Nöthnitzer Strasse 38, D01187 Dresden, Germany
[2]Semiconductor Physics Institute, LT-2600 Vilnius, Lithuania
(Received 6 October 2000)

Delayed feedback control of chaos is well known as an effective method for stabilizing unstable periodic orbits embedded in chaotic attractors. However, it had been shown that the method works only for a certain class of periodic orbits characterized by a finite torsion. Modification based on an unstable delayed feedback controller is proposed in order to overcome this topological limitation. An efficiency of the modified scheme is demonstrated for an unstable fixed point of a simple dynamic model as well as for an unstable periodic orbit of the Lorenz system.

DOI: 10.1103/PhysRevLett.86.2265 PACS numbers: 05.45.Gg

The problem of controlling chaos has attracted great interest among physicists over the past decade. As first pointed out by Ott, Grebogi, and Yorke [1], the existence of many unstable periodic orbits (UPOs) embedded in chaotic attractors raises the possibility of using very small external forces to obtain various types of regular behavior. Since that time a variety of chaos control methods have been developed [2], among which the delayed feedback control (DFC) scheme [3] has gained widespread acceptance. The DFC involves a control signal formed from the difference between the current state of the system and the state of the system delayed by one period of the UPO so that the control signal vanishes when the stabilization of the desired orbit is attained. The method does not require the real-time computer processing and is easy to apply in different experimental contexts. Examples of experimental implementation include electronic chaos oscillators [4], mechanical pendulums [5], lasers [6], a gas discharge system [7], chemical systems [8], and a cardiac system [9]. Several modifications [10,11] of the original DFC method have been proposed in order to improve its performance. Socolar *et al.* [10] have introduced an extended delayed feedback controller (EDFC) that involves the information from many previous states of the system. The EDFC has an advantage over the original method that it can stabilize periodic orbits with a greater degree of instability [12,13].

Stability analysis of the delayed feedback systems is very difficult. Nevertheless, some general analytical results have recently been obtained [14–17]. It has been shown that the DFC can stabilize only a certain class of periodic orbits characterized by a finite torsion. More precisely, the limitation is that any UPOs with an odd number of real Floquet multipliers (FMs) greater than unity can never be stabilized by the DFC. This statement was first proved by Ushio [14] for discrete time systems. Just *et al.* [15] and Nakajima [16] proved the same limitation for the continuous time DFC, and then this proof was extended for a wider class of delayed feedback schemes, including the EDFC [17]. Hence it seems hard to overcome this inherent limitation. So far only two efforts based on an oscillating feedback [18] and a half-period delay [19] have been taken to obviate this drawback. In both cases the mechanism of stabilization is rather unclear. Besides, the method of Ref. [19] is valid only for a special case of symmetric orbits. Here we report an unstable delayed feedback controller that can overcome the limitation without utilizing the symmetry of UPOs. The key idea is to artificially enlarge a set of real multipliers greater than unity to an even number by introducing into a feedback loop an unstable degree of freedom.

EDFC for $R > 1$.—First we illustrate the idea for a simple unstable discrete time system $y_{n+1} = \mu_s y_n$, $\mu_s > 1$ controlled by the EDFC:

$$y_{n+1} = \mu_s y_n - K F_n \,, \tag{1}$$

$$F_n = y_n - y_{n-1} + R F_{n-1} \,. \tag{2}$$

The free system $y_{n+1} = \mu_s y_n$ has an unstable fixed point $y^\star = 0$ with the only real eigenvalue $\mu_s > 1$ and, in accordance with the above limitation, cannot be stabilized by the EDFC for any values of the feedback gain K. This is so indeed if the EDFC is stable, i.e., if the parameter R in Eq. (2) satisfies the inequality $|R| < 1$. Only this case has been considered in the literature. However, it is easy to show that the unstable controller with the parameter $R > 1$ can stabilize this system. Using the ansatz $y_n, F_n \propto \mu^n$ one obtains the characteristic equation

$$(\mu - \mu_s)(\mu - R) + K(\mu - 1) = 0 \tag{3}$$

defining the eigenvalues μ of the closed loop system (1),(2). The system is stable if both roots $\mu = \mu_{1,2}$ of Eq. (3) are inside the unit circle of the μ complex plain, $|\mu_{1,2}| < 1$. Figure 1(a) shows the characteristic root-locus diagram for $R > 1$, as the parameter K varies from 0 to ∞. For $K = 0$, there are two real eigenvalues greater than unity, $\mu_1 = \mu_s$ and $\mu_2 = R$, which correspond to two independent subsystems (1) and (2), respectively; this means that both the controlled system and controller are unstable. With the increase of K, the eigenvalues approach each other on the real axes, then collide and pass to the complex plain. At $K = K_1 \equiv \mu_s R - 1$ they cross symmetrically the unite circle $|\mu| = 1$. Then both eigenvalues move

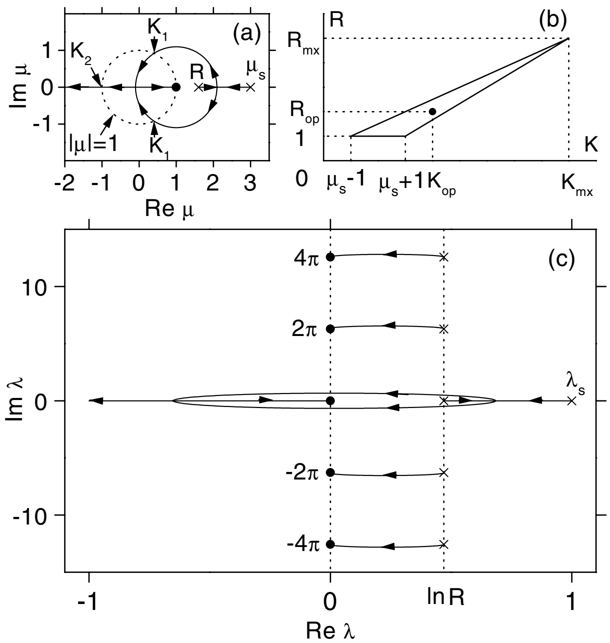

FIG. 1. Performance of (a),(b) discrete and (c) continuous EDFC for $R > 1$. (a) Root loci of Eq. (3) at $\mu_s = 3$, $R = 1.6$ as K varies from 0 to ∞. (b) Stability domain of Eqs. (1),(2) in the (K, R) plane; $K_{mx} = (\mu_s + 1)^2/(\mu_s - 1)$, $R_{mx} = (\mu_s + 3)/(\mu_s - 1)$. (c) Root loci of Eq. (6) at $\lambda_s = 1$, $R = 1.6$. The crosses and circles denote the location of roots at $K = 0$ and $K \to \infty$, respectively.

inside this circle, collide again on the real axes, and one of them leaves the circle at $K = K_2 \equiv (\mu_s + 1)(R + 1)/2$. In the interval $K_1 < K < K_2$, the closed loop system (1),(2) is stable. By a proper choice of the parameters R and K one can stabilize the fixed point with an arbitrarily large eigenvalue μ_s. The corresponding stability domain is shown in Fig. 1(b). For a given value μ_s, there is an optimal choice of the parameters $R = R_{\mathrm{op}} \equiv \mu_s/(\mu_s - 1)$, $K = K_{\mathrm{op}} \equiv \mu_s R_{\mathrm{op}}$ leading to zero eigenvalues, $\mu_1 = \mu_2 = 0$, such that the system approaches the fixed point in finite time.

It seems attractive to apply the EDFC with the parameter $R > 1$ for continuous time systems. Unfortunately, this idea fails. As an illustration, let us consider a continuous time version of Eqs. (1),(2)

$$\dot{y}(t) = \lambda_s y(t) - KF(t), \tag{4}$$

$$F(t) = y(t) - y(t - \tau) + RF(t - \tau), \tag{5}$$

where $\lambda_s > 0$ is the characteristic exponent of the free system $\dot{y} = \lambda_s y$ and τ is the delay time. By a suitable rescaling one can eliminate one of the parameters in Eqs. (4),(5). Thus, without a loss of generality we can take $\tau = 1$. Equations (4),(5) can be solved by the Laplace transform or simply by the substitution $y(t), F(t) \propto e^{\lambda t}$, which yields the characteristic equation:

$$1 + K \frac{1 - e^{-\lambda}}{1 - e^{-\lambda} R} \frac{1}{\lambda - \lambda_s} = 0. \tag{6}$$

In terms of the control theory, Eq. (6) defines the poles of the closed loop transfer function. The first and second fractions in Eq. (6) correspond to the EDFC and plant transfer functions, respectively. The closed loop system (4),(5) is stable if all the roots of Eq. (6) are in the left half-plane, $\mathrm{Re}\lambda < 0$. The characteristic root-locus diagram for $R > 1$ is shown in Fig. 1(c). When K varies from 0 to ∞, the EDFC roots move in the right half-plane from locations $\lambda = \ln R + 2\pi i n$ to $\lambda = 2\pi i n$ for $n = \pm 1, \pm 2 \ldots$. Thus, the continuous time EDFC with the parameter $R > 1$ has an infinite number of unstable degrees of freedom and many of them remain unstable in the closed loop system for any K.

Usual EDFC supplemented by an unstable degree of freedom.—Hereafter, we use the usual EDFC at $0 \le R < 1$; however, an additional unstable degree of freedom into a feedback loop is introduced. More specifically, for a dynamical system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, p)$ with a measurable scalar variable $y(t) = g[\boldsymbol{x}(t)]$ and an UPO of period τ at $p = 0$, we propose to adjust an available system parameter p by a feedback signal $p(t) = KF_u(t)$ of the following form:

$$F_u(t) = F(t) + w(t), \tag{7}$$

$$\dot{w}(t) = \lambda_c^0 w(t) + (\lambda_c^0 - \lambda_c^\infty) F(t), \tag{8}$$

$$F(t) = y(t) - (1 - R) \sum_{k=1}^{\infty} R^{k-1} y(t - k\tau), \tag{9}$$

where $F(t)$ is the usual EDFC described by Eq. (5) or equivalently by Eq. (9). Equation (8) defines an additional unstable degree of freedom with parameters $\lambda_c^0 > 0$ and $\lambda_c^\infty < 0$. We emphasize that whenever the stabilization is successful the variables $F(t)$ and $w(t)$ vanish, and thus vanishes the feedback force $F_u(t)$. We refer to the feedback law (7)–(9) as an unstable EDFC (UEDFC).

To get an insight into how the UEDFC works let us consider again the problem of stabilizing the fixed point

$$\dot{y} = \lambda_s y - KF_u(t), \tag{10}$$

where $F_u(t)$ is defined by Eqs. (7)–(9) and $\lambda_s > 0$. Here as well as in a previous example we can take $\tau = 1$ without a loss of generality. Now the characteristic equation reads

$$1 + KQ(\lambda) = 0, \tag{11}$$

$$Q(\lambda) \equiv \frac{\lambda - \lambda_c^\infty}{\lambda - \lambda_c^0} \frac{1 - e^{-\lambda}}{1 - e^{-\lambda} R} \frac{1}{\lambda - \lambda_s}. \tag{12}$$

The first fraction in Eq. (12) corresponds to the transfer function of an additional unstable degree of freedom. Root loci of Eq. (11) are shown in Fig. 2. The poles and zeros of Q function define the value of roots at $K = 0$ and $K \to \infty$, respectively. Now at $K = 0$, the EDFC roots $\lambda = \ln R + 2\pi i n$, $n = 0, \pm 1, \ldots$ are in the left half-plane. The only root λ_c^0 associated with an additional unstable degree of freedom is in the right half-plane. That root and the root λ_s of the fixed point collide on the real axes, pass to the complex plane, and at $K = K_1$ cross into the left half-plane.

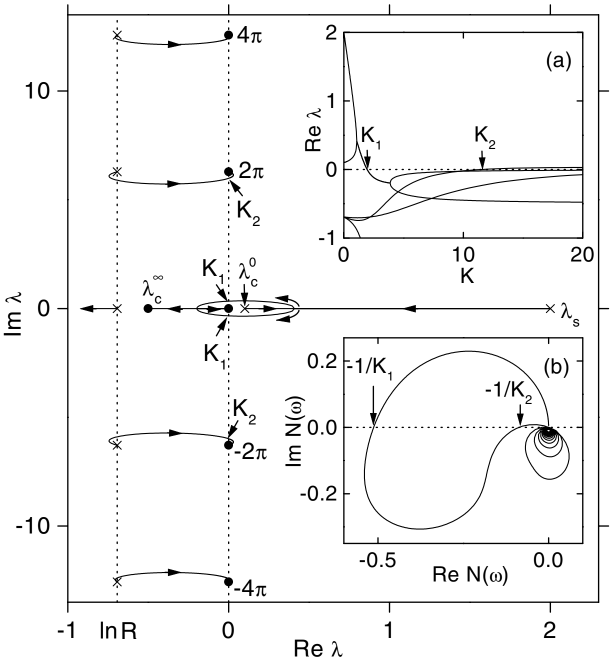

FIG. 2. Root loci of Eq. (11) at $\lambda_s = 2$, $\lambda_c^0 = 0.1$, $\lambda_c^\infty = -0.5$, $R = 0.5$. Insets (a) and (b) show Reλ vs K and the Nyquist plot, respectively. The boundaries of the stability domain are $K_1 \approx 1.95$ and $K_2 \approx 11.6$.

For $K_1 < K < K_2$, all roots of Eq. (11) satisfy the inequality Re$\lambda < 0$, and the closed loop system (7)–(10) is stable. The stability is destroyed at $K = K_2$ when the EDFC roots $\lambda = \ln R \pm 2\pi i$ in the second "Brillouin zone" cross into Re$\lambda > 0$. The dependence of the five largest Reλ on K is shown in inset (a) of Fig. 2. Inset (b) shows the Nyquist plot, i.e., a parametric plot Re$N(\omega)$ versus Im$N(\omega)$ for $\omega \in [0, \infty]$, where $N(\omega) \equiv Q(i\omega)$. The Nyquist plot provides the simplest way of determining the stability domain; it crosses the real axes at Re$N = -1/K_1$ and Re$N = -1/K_2$.

As a more involved example let us consider the Lorenz system under the UEDFC:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} -\sigma x + \sigma y \\ rx - y - xz \\ xy - bz \end{pmatrix} - KF_u(t) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}. \qquad (13)$$

We assume that the output variable is y and the feedback force $F_u(t)$ [Eqs. (7)–(9)] perturbs only the second equation of the Lorenz system. Denote the variables of the Lorenz system by $\boldsymbol{\rho} = (x, y, z)$ and those extended with the controller variable w by $\boldsymbol{\xi} = (\boldsymbol{\rho}, w)^T$. For the parameters $\sigma = 10$, $r = 28$, and $b = 8/3$, the free ($K = 0$) Lorenz system has a period-one UPO, $\boldsymbol{\rho}_0(t) \equiv (x_0, y_0, z_0) = \boldsymbol{\rho}_0(t + \tau)$, with the period $\tau \approx 1.5586$ and all real FMs: $\mu_1 \approx 4.714$, $\mu_2 = 1$, and $\mu_3 \approx 1.19 \times 10^{-10}$. This orbit cannot be stabilized by the usual DFC or EDFC, since only one FM is greater than unity. The ability of the UEDFC to stabilize this orbit can be verified by a linear analysis of Eqs. (13) and (7)–(9). Small deviations $\delta\boldsymbol{\xi} = \boldsymbol{\xi} - \boldsymbol{\xi}_0$ from the periodic solution $\boldsymbol{\xi}_0(t) \equiv (\boldsymbol{\rho}_0, 0)^T = \boldsymbol{\xi}_0(t + \tau)$ may be decomposed into eigenfunctions according to the Floquet theory, $\delta\boldsymbol{\xi} = e^{\lambda t}\boldsymbol{u}$, $\boldsymbol{u}(t) = \boldsymbol{u}(t + \tau)$, where λ is the Floquet exponent. The Floquet decomposition yields linear periodically time dependent equations $\delta\dot{\boldsymbol{\xi}} = A\delta\boldsymbol{\xi}$ with the boundary condition $\delta\boldsymbol{\xi}(\tau) = e^{\lambda\tau}\delta\boldsymbol{\xi}(0)$, where

$$A = \begin{pmatrix} -\sigma & \sigma & 0 & 0 \\ r - z_0(t) & -(1 + KH) & -x_0(t) & -K \\ y_0(t) & x_0(t) & -b & 0 \\ 0 & (\lambda_c^0 - \lambda_c^\infty)H & 0 & \lambda_c^0 \end{pmatrix}. \qquad (14)$$

Because of equality $\delta y(t - k\tau) = e^{-k\lambda\tau}\delta y(t)$, the delay terms in Eq. (9) are eliminated, and Eq. (9) is transformed to $\delta F(t) = H\delta y(t)$, where

$$H = H(\lambda) = (1 - e^{-\lambda\tau})/(1 - e^{-\lambda\tau}R) \qquad (15)$$

is the transfer function of the EDFC. The price for this simplification is that the Jacobian A, defining the exponents λ, depends on λ itself. The eigenvalue problem may be solved with an evolution matrix Φ_t that satisfies

$$\dot{\Phi}_t = A\Phi_t, \qquad \Phi_0 = I. \qquad (16)$$

The eigenvalues of Φ_τ define the desired exponents:

$$\det[\Phi_\tau(H) - e^{\lambda\tau}I] = 0. \qquad (17)$$

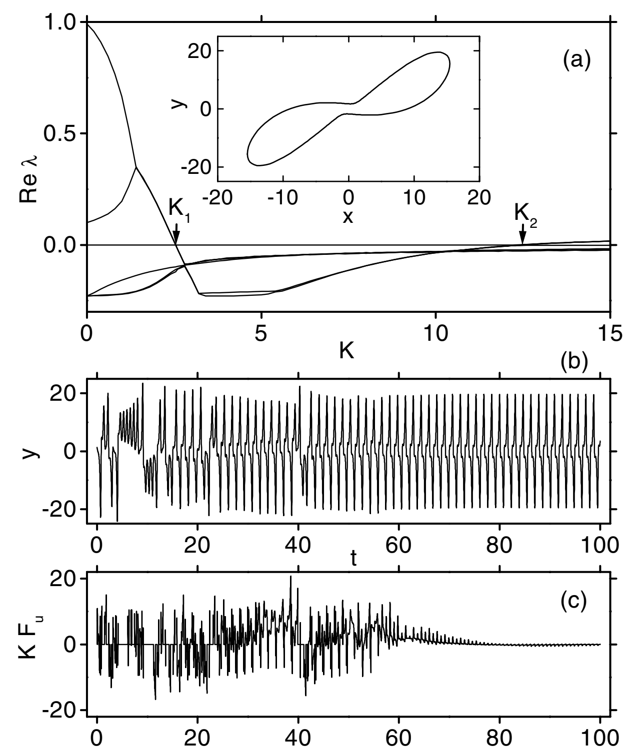

FIG. 3. Stabilizing an UPO of the Lorenz system. (a) Six largest Reλ vs K. The boundaries of the stability domain are $K_1 \approx 2.54$ and $K_2 \approx 12.3$. The inset shows the (x, y) projection of the UPO. (b) and (c) shows the dynamics of $y(t)$ and $F_u(t)$ obtained from Eqs. (13),(7)–(9). The parameters are $\lambda_c^0 = 0.1$, $\lambda_c^\infty = -2$, $R = 0.7$, $K = 3.5$, $\varepsilon = 3$, $\lambda_r = 10$.

We emphasize the dependence Φ_τ on H conditioned by the dependence of A on H. Thus by solving Eqs. (15)–(17), one can define the Floquet exponents λ (or multipliers $\mu = e^{\lambda\tau}$) of the Lorenz system under the UEDFC. Figure 3(a) shows the dependence of the six largest Reλ on K. There is an interval $K_1 < K < K_2$, where the real parts of all exponents are negative. Basically, Fig. 3(a) shows the results similar to those presented in Fig. 2(a). The unstable exponent λ_1 of an UPO and the unstable eigenvalue λ_c^0 of the controller collide on the real axes and pass into the complex plane providing an UPO with a finite torsion. Then this pair of complex conjugate exponents cross into domain Re$\lambda < 0$, just as they do in the simple model of Eq. (10).

Direct integration of the nonlinear Eqs. (13),(7)–(9) confirms the results of linear analysis. Figures 3(b) and 3(c) show a successful stabilization of the desired UPO with an asymptotically vanishing perturbation. In this analysis, we used a restricted perturbation similar as we did in Ref. [3]. For $|F(t)| < \varepsilon$, the control force $F_u(t)$ is calculated from Eqs. (7)–(9); however, for $|F(t)| > \varepsilon$, the control is switched off, $F_u(t) = 0$, and the unstable variable w is dropped off by replacing Eq. (8) with the relaxation equation $\dot{w} = -\lambda_r w$, $\lambda_r > 0$.

To verify the influence of fluctuations a small white noise with the spectral density $S(\omega) = a$ has been added to the right-hand side of Eqs. (8),(13). At every step of integration the variables x, y, z, and w were shifted by an amount $\sqrt{12ha}\,\xi_i$, where ξ_i are the random numbers uniformly distributed in the interval $[-0.5, 0.5]$ and h is the stepsize of integration. The control method works when the noise is increased up to $a \approx 0.02$. The variance of perturbation increases proportionally to the noise amplitude, $\langle F_u^2(t)\rangle = ka$, $k \approx 17$. For a large noise $a > 0.02$, the system intermittently loses the desired orbit.

In conclusion, an unstable degree of freedom introduced into a feedback loop can overcome the well known limitation of the delayed feedback scheme. The proposed unstable controller can stabilize unstable periodic orbits with a zero torsion and can be used for a wider class of chaotic systems, e.g., the Lorenz system.

The author thanks W. Just and A. Kittel for fruitful discussions and H. Kantz for the invitation to the Max-Planck-Institut für Physik komplexer Systeme and collaboration.

*Electronic address: pyragas@kes0.pfi.lt

[1] E. Ott, C. Grebogi, and J. A. Yorke, Phys. Rev. Lett. **64**, 1196 (1990).
[2] T. Shinbrot, C. Grebogi, E. Ott, and J. A. Yorke, Nature (London) **363**, 411 (1993); *Handbook of Chaos Control,* edited by H. G. Shuster (Wiley-VCH, Weiheim, 1999).
[3] K. Pyragas, Phys. Lett. A **170**, 421 (1992).
[4] K. Pyragas and A. Tamaševičius, Phys. Lett. A **180**, 99 (1993); A. Kittel, J. Parisi, K. Pyragas, and R. Richter, Z. Naturforsch. A **49**, 843 (1994); D. Gauthier, D. W. Sukow, H. M. Concannon, and J. E. S. Socolar, Phys. Rev. E **50**, 2343 (1994).
[5] T. Hikihara and T. Kawagoshi, Phys. Lett. A **211**, 29 (1996); D. J. Christini, V. In, M. Spano, W. Ditto, and J. J. Collins, Phys. Rev. E **56**, R3749 (1997).
[6] S. Bielawski, D. Derozier, and P. Glorieux, Phys. Rev. E **49**, R971 (1994); W. Lu, D. Yu, and R. G. Harrison, Int. J. Bifurcation Chaos Appl. Sci. Eng. **8**, 1769 (1998).
[7] Th. Mausbach, Th. Klinger, A. Piel, A. Atipo, Th. Pierre, and G. Bonhomme, Phys. Lett. A **228**, 373 (1997).
[8] P. Parmananda, R. Madrigal, M. Rivera, L. Nyikos, Z. Kiss, and V. Gaspar, Phys. Rev. E **59**, 5266 (1999); A. P. M. Tsui and A. J. Jones, Physica (Amsterdam) **135D**, 41 (2000).
[9] K. Hall, D. J. Christini, M. Tremblay, J. J. Collins, L. Glass, and J. Billette, Phys. Rev. Lett. **78**, 4518 (1997).
[10] J. E. S. Socolar, D. W. Sukow, and D. J. Gauthier, Phys. Rev. E **50**, 3245 (1994).
[11] A. Kittel, J. Parisi, and K. Pyragas, Phys. Lett. A **198**, 433 (1995).
[12] K. Pyragas, Phys. Lett. A **206**, 323 (1995).
[13] M. E. Bleich and J. E. S. Socolar, Phys. Lett. A **210**, 87 (1996).
[14] T. Ushio, IEEE Trans. Circuits Syst. I, Fundam. Theory Appl. **43**, 815 (1996).
[15] W. Just, T. Bernard, M. Ostheimer, E. Reibold, and H. Benner, Phys. Rev. Lett. **78**, 203 (1997).
[16] H. Nakajima, Phys. Lett. A **232**, 207 (1997).
[17] H. Nakajima and Y. Ueda, Physica (Amsterdam) **111D**, 143 (1998).
[18] S. Bielawski, D. Derozier, and P. Glorieux, Phys. Rev. A **47**, R2492 (1993); H. G. Schuster and M. P. Stemmler, Phys. Rev. E **56**, 6410 (1997).
[19] H. Nakajima and Y. Ueda, Phys. Rev. E **58**, 1757 (1998).