Conditional Lyapunov exponents from time series

K. Pyragas

Semiconductor Physics Institute, LT-2600 Vilnius, Lithuania
(Received 9 May 1997)

A method for estimating conditional Lyapunov exponents from time series of two unidirectionally coupled chaotic systems is developed. It uses two scalar data sets, one taken from the driving and the other from the response system, and enables one to detect a generalized synchronization in an experiment without recourse to an auxiliary response system. The method is illustrated on coupled maps as well as coupled chaotic flow models. [S1063-651X(97)02311-8]

PACS number(s): 05.45.+b

I. INTRODUCTION

The cooperative behavior of coupled chaotic systems has attracted considerable attention lately. Synchronization effects are observed in many physical and biological processes and they are responsible for the transition to low-dimensional attractors in systems with many degrees of freedom. Synchronization of chaos is often understood as a behavior in which two coupled systems exhibit identical chaotic oscillations in the same way as identical synchronization occurs the asymptotic dynamics of the response system is independent of its initial conditions and is completely determined by the driving system. Geometrically, such synchronization occurs if there exists a map \( F: X \rightarrow Y \) that takes the trajectories of the attractor in the driving space \( X = \{x_1, x_2, \ldots, x_D\} \) into the trajectories of the response space \( Y = \{y_1, y_2, \ldots, y_D\} \) so that \( Y(t) = \Phi(X(t)) \) and if this map does not depend upon initial conditions of the response system [4]. When \( \Phi \) differs from the identity the detection of the GS in an experiment is a difficult task. One way to recognize the GS is to construct an auxiliary response system \( Y' \) identical to \( Y \), link it to the driving system \( X \) in the same way as \( Y \) is linked to \( X \).

\[
\dot{X} = F(X), \quad \dot{Y} = G(Y,X),
\]

\[
\dot{Y}' = G(Y',X),
\]

and check the existence of the IS between \( Y \) and \( Y' \). If such synchronization occurs the asymptotic dynamics of the response system is independent of its initial conditions and is completely determined by the driving system. Geometrically, this implies a collapse of the overall evolution onto a stable synchronization manifold \( M = \{X,Y\} : \Phi(X) = Y \) in the full phase space of two systems \( X \oplus Y \) and so leads to a functional relationship between \( X \) and \( Y \) variables defining the GS [5,6].

Unfortunately, the auxiliary system approach is of limited utility. The method fails for systems whose dynamical equations are not available. Even though the dynamical equations are known (e.g., in electronic circuit experiments [5]), the auxiliary response system can be designed only with finite accuracy; it cannot be an exact copy of the original response system. An alternative approach to detect the GS is to estimate the conditional Lyapunov exponents (CLEs) \( \lambda^R_1 \geq \lambda^R_2 \geq \cdots \geq \lambda^R_d \) from observed time series. The CLEs define the stability of both the identity manifold \( Y' = Y \) in \( X \oplus Y \) phase space and the synchronized manifold \( Y = \Phi(X) \) in \( X \oplus Y \) space [5] and are determined by the variational equation of the response system at \( \delta X = 0 \),

\[
\delta \dot{Y} = D_Y G(Y,X) \delta Y,
\]

where \( D_Y G \) denotes the Jacobian matrix with respect to the \( Y \) variable. The condition of GS is \( \lambda^R_1 < 0 \).

Note that for systems with a skew product structure described by Eq. (1), the CLEs represent a part of the whole Lyapunov spectrum \( \lambda_1, \lambda_2, \ldots, \lambda_{r+d} \) of this system. The remainder of this spectrum consists of Lyapunov exponents \( \lambda^D_1 \geq \lambda^D_2 \geq \cdots \geq \lambda^D_d \) of the driving system (1a). In other words, to obtain the whole spectrum of Lyapunov exponents of system (1) in usual (descending) order, \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{r+d} \), the combined spectrum of the driving Lyapunov exponents and the CLEs \( \lambda^R_1, \lambda^R_2, \ldots, \lambda^R_d, \lambda^D_1, \lambda^D_2, \ldots, \lambda^D_d \) has to be resorted in order of their numerical size. If the whole spectrum of the Lyapunov exponents is known then one can extract information about the properties of the synchronization manifold.

Depending on the properties of the map \( \Phi \), the GS can be subdivided into two types: weak synchronization (WS) and strong synchronization (SS) [7]. The WS is associated with the continuous \( C^0 \) but nonsmooth map \( \Phi \) so that the synchronization manifold \( M = \{(X,Y) : \Phi(X) = Y\} \) has a fractal structure and the global dimension \( d_G \) of the strange attractor in the whole phase space \( X \oplus Y \) is larger than the dimension of the driving attractor \( d_D \) in \( X \) subspace, \( d_G > d_D \). The SS is related to the smooth map \( \Phi \) with the degree of smoothness \( C^1 \) or higher when the response system does not have an

*Electronic address: pyragas@kes0.pfi.lt

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effect on the global dimension, i.e., \(d_G=d_D\) [8]. This is valid, for example, for the IS, which is a particular case of the SS.

In most cases dimensions \(d_G\) and \(d_D\) can be estimated from the Kaplan-Yorke conjecture [9]

\[
d_G=1 + \frac{1}{\lambda_{i_G+1}} \sum_{i=1}^{i_G} \lambda_i, \tag{4}
\]

\[
d_D=1 + \frac{1}{\lambda_{i_D+1}} \sum_{i=1}^{i_D} \lambda_i, \tag{5}
\]

where \(i_G\) and \(i_D\) are the largest integers for which the corresponding sums over \(i\) are non-negative. The global Lyapunov dimension is independent of the response system \((d_G=d_D)\) at the condition [7] \(\lambda_1^S<\lambda_1^D\). If this condition is fulfilled and relations (4) and (5) are valid, we have the SS.

Note that only a finite number of Lyapunov exponents can be reliably determined from data on the attractor [10]. An appropriate cutoff value for the number of exponents is related to the global Lyapunov dimension and is equal to \(i_G+1\). The only exponents that are included in Eq. (4) are fundamentally important to the character of the attractor and their estimation is available from time series. In the case of WS at least a maximal CLE affects the global dimension and hence can be estimated from time series. The condition of strong synchronization \(\lambda_1^S<\lambda_1^D\) corresponds to the case when the global dimension \(d_G\) does not depend on CLEs. Thus we cannot expect a reliable estimation of CLEs from time series above the threshold of SS. However, the CLEs can be determined just before this threshold and this suffices to estimate characteristic values of control parameters corresponding to the onset of SS.

II. ALGORITHM

Suppose that an experimental system under investigation can be simulated by Eqs. (1). We imagine that the equations are unknown, but two scalar time series \(x_i\) and \(y_i\), \(i=1,...,N\), corresponding to the driving and response subsystems, respectively, are available for observation. We assume that the time interval \(\tau\) between measurements is fixed so that \(x_i=x(i\tau)\) and \(y_i=y(i\tau)\). Below \(\tau\) is identified with the delay time of phase-space reconstruction in step (a) of our algorithm. In principle, any choice of \(\tau\) is acceptable in the limit of an infinite amount of data. For a small amount of data, the choice of \(\tau\) can be based, for example, on the evaluation of mutual information [11].

Due to the unidirectional coupling the \(x_i \) series does not contain any information about the response system, while the \(y_i \) series does contain the information about both subsystems. Since the CLEs represent a part of the whole Lyapunov spectrum, one can expect that they can be determined by the standard algorithms [10,12,13] from \(y_i \) time series. However, the CLEs may be placed far from the maximal Lyapunov exponent in the whole spectrum ordered in descending fashion, while the standard algorithms give reliable values only for a few largest exponents [10,12,13]. Moreover, there is a nontrivial problem to define which exponents belong to the CLEs and which to the driving system, even though the whole spectrum of the Lyapunov exponents is reliably determined. These problems can be solved in the framework of the algorithm involving information from both scalar time series \(x_i\) and \(y_i\). Here we mainly use the ideas of the algorithm proposed by Eckmann et al. [13] based on the construction of local linear maps. The mappings with a higher order of Taylor series [10] are beyond our scope. We extend the Eckmann-Kamphorst-Ruelle-Ciliberto (EKRC) algorithm for the case of two time series and adopt it for the direct estimation of the CLEs. The reliability of estimating the maximal CLE by our algorithm is comparable to that of estimating the conventional maximal Lyapunov exponent by the EKRC algorithm. A copy of the computer program implementing this algorithm can be obtained from the author.

To speed up the computation and to bring our consideration closer to a real experimental situation we present the time series \(x_i\) and \(y_i\) by integer numbers normed to the same maximal value \(M_0\) so that \(0\leq x_i \leq M_0\) and \(0\leq y_i \leq M_0\). Typically we take \(M_0=10000\) in accordance with a precision of \(10^{-4}\). Similarly to the EKRC algorithm, our algorithm involves the following three steps: (a) reconstructing the dynamics by the time-delay method [14] and finding the neighbors of the fiducial trajectory, (b) obtaining the tangent maps by a least-squares fit, and (c) deducing the CLEs from the tangent maps. Now we consider these steps in detail.

(a) We choose different embedding dimensions \(E_x\) and \(E_y\) for the driving and response systems and define \((E_x+E_y)\)-dimensional vectors

\[
R_i=\{x_{i-E_x+1},...,x_{i-2},x_i,y_{i-E_y+1},...,y_{i-2},y_i\} \tag{6}
\]

for \(i=i_0=\max(E_x,E_y)\), \(i_0+1,...,N\), to construct the dynamics of the fiducial trajectory in the whole \(X\oplus Y\) phase space. In view of step (b) we have to determine the neighbors of \(R_i\), i.e., the points \(R_j\) of the orbit that are contained in a ball of small radius \(\epsilon_i\) centered at \(R_i\),

\[
\|R_j-R_i\|<\epsilon_i. \tag{7}
\]

Here \(\|\|\) implies the maximal projection of the vector rather than the Euclidean norm. This allows a fast search for the \(R_j\) by first sorting the data [13]. Denote by \(J_i\) the number of neighbors \(R_j\) of \(R_i\) within a distance \(\epsilon_i\), as determined by Eq. (7). Clearly, \(J_i\) depends on \(\epsilon_i\). In (b) we discuss the choice of these parameters for every \(i\).

(b) Having embedded our dynamical system, we want to determine the tangent map that describes how the time evolution sends small vectors around \(R_i=\{x_i, Y_i\}\) to small vectors around \(Y_{i+m}\). This problem can be considered in the phase space of reduced dimension [13]. Following Ref. [10], we introduce the local dimensions \(L_x\approx E_x\) and \(L_y\approx E_y\) that reflect the number of dimensions necessary to capture the geometry of a small neighborhood of the attractor after it has been successfully embedded (i.e., the time-delay representation is diffeomorphic to the original attractor). Dimensions \(L_x\) and \(L_y\) are used for constructing the local maps and correspond to the number of Lyapunov exponents of the driving system and CLEs, respectively, produced by algorithm. The transition from embedding dimensions to local dimensions is
performed similarly to that in Ref. [13]. We drop the intermediate components in Eq. (6) and define the \(L_x\)-dimensional \(X_i\) and \(L_y\)-dimensional \(Y_i\) vectors as
\[
X_i = (x_{i-E_1+1}, \ldots, x_{i-m}, x_i)^T, \\
Y_i = (y_{i-E_1+1}, \ldots, y_{i-m}, y_i)^T.
\]
(8a, 8b)

The dimensions \(L_x \leq E_x\) and \(L_y \leq E_y\) are determined by equalities \(E_x = (L_x - 1)m + 1\) and \(E_y = (L_y - 1)m + 1\), which we assume to hold for some integer \(m \geq 1\). The case \(m = 1\) corresponds to \(L_x = E_x\), \(L_y = E_y\). When \(m > 1\) the dimension of the tangent map is reduced with respect to the embedding dimension and this can help to avoid the spurious Lyapunov exponents [13].

The tangent map is defined by two matrices \(A_i\) and \(B_i\), which are obtained by looking for neighbors \(J_i\) of \(R_i\) and imposing
\[
A_i(X_j - X_i) + B_i(Y_j - Y_i) = Y_{j+m} - Y_{i+m}.
\]
(9)

\(A_i\) is the rectangular \(L_x \times L_y\) matrix and \(B_i\) is the square \(L_y \times L_y\) matrix, which in view of Eqs. (8) and (9) have the form
\[
A_i = \begin{pmatrix}
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{pmatrix}
\]
\[
B_i = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{pmatrix}
\]

Matrix \(A_i\) contains \(L_x\) unknown elements \(a_{i}^j\), \(k = 1, 2, \ldots, L_x\), and matrix \(B_i\) contains \(L_y\) unknowns \(b_{i}^j\), \(k = 1, 2, \ldots, L_y\). These \(L_x + L_y\) unknowns are obtained by a least-squares fit
\[
\min_{a_{i}^j,b_{i}^j} \frac{1}{J_i} \sum_{j=1}^{J_i} \| A_i(X_j - X_i) + B_i(Y_j - Y_i) - (Y_{j+m} - Y_{i+m}) \|_2^2,
\]

where \(\| \cdot \|_2^2\) denotes the square of the Euclidean norm of the vector. This problem reduces to a set of \(L_x + L_y\) linear equations with respect to \(L_x + L_y\) unknowns \(a_{i}^j, b_{i}^j\), which we solve by the LU decomposition algorithm [15]. Obviously, this algorithm fails if the number of neighbors \(J_i\) of the fiducial point \(R_i\) is less than the number of unknowns, \(J_i < L_x + L_y\). To avoid this problem the radius \(\epsilon_i\) has to be chosen to be sufficiently large. For the specific examples discussed below we have selected \(\epsilon_i\) and \(J_i\) as follows. Count the number of neighbors \(J_i\) of \(R_i\) corresponding to increasing values of \(\epsilon_i\) from a preselected sequence of possible values and stop when \(J_i\) exceeds for the first time \(J_{\text{min}} = 2(L_x + L_y)\). To speed up the calculations we also stop the search for the neighbors when for given \(\epsilon_i\) the number of neighbors exceeds some maximal value \(J_{\text{max}} = 40\). Thus, for every \(i, J_i\) is in the interval \([J_{\text{min}}, J_{\text{max}}]\).

(c) Step (b) gives matrices \(A_i\) and \(B_i\) of the tangent map, which represent the reconstructed Jacobians \(D_x G\) and \(D_y G\) of Eq. (1b) with respect to \(X\) and \(Y\) variables, respectively. The CLEs are determined by the product of the matrices \(B_i B_i + B_i B_i + \cdots\). To extract the CLEs from this product we use the QR decomposition technique [13,15]. The method recursively defines an orthogonal matrix \(Q_l\) and an upper triangular matrix \(R_l\), \(l = 0, 1, \ldots, L - 1\), via \(B_i B_i + B_i B_i + \cdots = Q_l R_l + R_l + 1\), where \(Q_0\) is the unit matrix. The CLEs are given by
\[
\lambda_{n}^{R} m = \frac{1}{L} \sum_{l=0}^{L-1} \ln(Q_{l})_{nn},
\]
where \(K < (N - i_0)/m\) is the available number of matrices and \((Q_{l})_{nn}\) is the diagonal element of the matrix \(Q_l\). Note that in final step we do not require a knowledge of the matrix \(A_i\). However, the use of this matrix in step (b) is necessary in order to determine correctly the tangent map (9) and hence the matrix \(B_i\) defining the CLEs.

Now we illustrate our algorithm with two specific examples.

III. EXAMPLES

A. Coupled Hénon maps

The first example represents two identical unidirectionally coupled Hénon [16] maps
\[
\begin{align*}
\dot{x}_1(i+1) &= f(x_1(i), x_2(i)), \\
\dot{x}_2(i+1) &= b x_1(i), \\
\dot{y}_1(i+1) &= (1 - k) f(y_1(i), y_2(i)) + k f(x_1(i), x_2(i)), \\
\dot{y}_2(i+1) &= b y_1(i),
\end{align*}
\]
(10a, 10b)

where \(f(x_1, x_2) = 1 - a x_1^2 + x_2, a = 1.4, b = 0.3\), and \(k\) is the control parameter defining the coupling strength. At any \(k\), this system has an invariant manifold \(Y = X\) and hence admits the IS, which in this case is equivalent to the SS. The IS appears when the manifold \(Y = X\) becomes stable. This happens when \(k\) exceeds some threshold \(k > k_3 = 0.34\) so that the transverse Lyapunov exponents of the manifold \(Y = X\) become negative. Before reaching this threshold, the system exhibits the GS in the form of WS. This is observed in the interval of parameter \(k \in \{k_1, k_2\}\), with \(k_1 = 0.16\) and \(k_2 = 0.20\). Here the maximal CLE is negative, while the maximal transverse Lyapunov exponent of the identity manifold \(Y = X\) is positive. This means that systems \(Y\) and \(Y'\) [here by \(Y'\) we imply the auxiliary response system constructed in accordance with Eq. (10b)] are synchronized in the sense of IS and there is no IS between \(X\) and \(Y\).

To test the algorithm two scalar time series \(x_1(i)\) and \(y_1(i)\) were treated as experimental data. The results presented in Table I correspond to a fixed value \(k = 0.1\) and different values of local dimensions \(L_x\) and \(L_y\). For comparison, we calculated the correct values of CLEs \(\lambda_{n}^{R} \approx 0.227\) and
TABLE I. CLEs for coupled Hénon maps at $k=0.1$ computed from $N=50$ 000 data points evaluated with the sampling time $\tau = 1$. We vary the local dimensions $L_x$ and $L_y$ at fixed $m=1$ so that they coincide with the embedding dimensions, $E_x=L_x$ and $E_y=L_y$. The correct values of CLEs calculated directly from Eqs. (10) are $\lambda_1^R=0.227$, $\lambda_2^R=1.537$. For $L_y>2$, the algorithm gives $L_y=2$ spurious CLEs in parallel with two valid CLEs. The values corresponding to the valid CLEs are underlined.

<table>
<thead>
<tr>
<th>$L_y$</th>
<th>$L_x$</th>
<th>$\lambda_1^R$</th>
<th>$\lambda_2^R$</th>
<th>$\lambda_3^R$</th>
<th>$\lambda_4^R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2</td>
<td>0.228</td>
<td>−1.408</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>0.274</td>
<td>−1.411</td>
<td></td>
<td></td>
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<tr>
<td>2</td>
<td>4</td>
<td>0.219</td>
<td>−1.402</td>
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<td></td>
</tr>
<tr>
<td>3</td>
<td>2</td>
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<td>0.203</td>
<td>−1.558</td>
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</tr>
<tr>
<td>3</td>
<td>3</td>
<td>0.459</td>
<td>0.186</td>
<td>−1.547</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>0.489</td>
<td>0.178</td>
<td>−1.546</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>0.530</td>
<td>0.206</td>
<td>−0.962</td>
<td>−1.629</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>0.512</td>
<td>0.189</td>
<td>−0.863</td>
<td>−1.612</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>0.536</td>
<td>0.191</td>
<td>−0.786</td>
<td>−1.613</td>
</tr>
</tbody>
</table>

$\lambda_2^R \approx −1.537$ ($k=0.1$) using Eqs. (10). For any $L_x \geq 2$ and $L_y \geq 2$, the algorithm gives two CLEs close to these correct values. If $L_y$ is chosen correctly [i.e., equal to the dimension of the response system (10b) $L_y=r=2$] we obtain the right number of CLEs whose values weakly depend on $L_x$ provided $L_x \geq 2$. For $L_y>2$, the algorithm gives spurious CLEs in parallel with valid CLEs.

One way of identifying spurious exponents is to analyze the influence of external noise [10]. This is illustrated in Fig. 1. We have added Gaussian white noise to the data points with the standard deviation $\sigma$. In Fig. 1(a) we have used $L_x=L_y=2$, while in Fig. 1(b) we used $L_x=L_y=3$, which gives one spurious CLE. The spurious CLE in Fig. 1(b) drops rapidly as the added noise is increased, going from $+0.7$ down to $−0.9$.

Figure 2 shows a correlation between the dependence of CLEs on the coupling strength $k$ estimated from time series with that calculated directly from Eqs. (10). Good agreement, especially for the maximal CLE, is observed for $k < k_1$. For $k > k_1$, we have the SS with identical time series $y_1(i) = x_1(i)$ and the algorithm fails. This is in agreement with the general prediction that CLEs cannot be reliably estimated from time series in the domain of SS. However, the algorithm gives the correct values of the maximal CLE in the immediate vicinity of the threshold $k \approx k_1$.

B. Coupled Rössler and Lorenz systems

As a second example we choose the model considered in Ref. [7], which illustrates the GS in essentially different time-continuous systems. It represents unidirectionally coupled Rössler [17] and Lorenz [18] equations

$$\begin{align*}
\frac{dx_1}{dt} &= \alpha (−x_2−x_3) + x_1 + 2x_2, \\
\frac{dx_2}{dt} &= 0.2 + x_1 x_3 − 5.7 x_3, \\
\frac{dx_3}{dt} &= −10 (y_1 + y_2) + k x_2.
\end{align*}$$

(11a)

$$\begin{align*}
\frac{dy_1}{dt} &= 10 (−y_1 + y_2), \\
\frac{dy_2}{dt} &= 28y_1 − 3y_1 y_3 + k x_2, \\
\frac{dy_3}{dt} &= y_1 y_2 − 8/3 y_3.
\end{align*}$$

(11b)

Here Eqs. (11a) and (11b) correspond to the Rössler (driving) and the Lorenz (response) system, respectively. The multiplier $\alpha = 6$ is introduced to control the time scale of the driving system. The last term in Eq. (11b) describes the coupling, where $k$ is the coupling strength. Despite the lack of any symmetry admitting the IS, this system exhibits the GS [7].
were treated as experimentally available outputs. In Figs. 3(a) and 3(b), the calculated maximal CLE and the global Lyapunov dimension \(d_G\) were shown as functions of the coupling strength \(k\) for coupled R"ossler and Lorenz systems at \(N=50,000\), \(\tau=0.15\) for \(k\leq 10\), and \(\tau=0.03\) for \(k>10\). The local dimensions \(L_x=L_y=3\) and \(m=1\). To estimate \(d_G\) one needs a knowledge of the Lyapunov exponents of the driving system, \(\lambda_0^{D}=0.41\), \(\lambda_2^{D}=0.00\), and \(\lambda_3^{D}=-37.66\). We evaluated them by a standard EKRK algorithm from \(x_1(t)\) time series. The threshold of GS \(k_1=6.66\) corresponds to \(\lambda_1^{R}(k_1)=0\).

The global dimension \(d_G\) saturates to the value approximately equal to the driving dimension \(d_D=2+\lambda_1^{D}/|\lambda_3^{D}|\approx2.01\) at \(k>40\). In this domain, the synchronization manifold is almost smooth. For comparison, the correct characteristics \(\lambda_1^{R}(k)\) and \(d_G(k)\) determined directly from Eqs. (11) are shown by dotted lines.

When testing the algorithm, the variables \(x_1(t)\) and \(y_1(t)\) were treated as experimentally available outputs. In Figs. 3(a) and 3(b), the calculated maximal CLE and the global Lyapunov dimension, respectively, are shown as functions of coupling strength \(k\). For comparison, the same characteristics determined directly from Eqs. (11) are presented. Good agreement of corresponding characteristics is observed in a large interval of coupling strengths. These results allow us to estimate both the threshold of GS and the smoothness of the synchronization manifold. The threshold of GS is obtained from \(\lambda_1^{R}(k_1)=0\) and is approximately equal to \(k_1=6.66\). In the case of the driving system presented by a three-dimensional flow the condition of SS becomes \(\lambda_1^{R}(k)<\lambda_3^{D}\).

For the system of equations (11a), we have \(\lambda_2^{D}=0.41\), \(\lambda_2^{R}
qr0.00\), and \(\lambda_3^{R}=-37.66\) and the driving Lyapunov dimension is equal to \(d_D=2+\lambda_1^{D}/|\lambda_3^{D}|\approx2.01\). Because of the large negative value of \(\lambda_3\) the condition \(\lambda_1^{R}(k)<\lambda_3^{R}\) is not achieved even for very large \(k\approx1000\). Thus we have the WS for all \(k>k_1\) and the algorithm of estimating CLEs from time series works well in the whole considered interval of \(k\).

An algorithm for estimating conditional Lyapunov exponents from two scalar time series, one taken from the driving \(X\) and the other from the response \(Y\), is suggested. This analysis of experimental data enables one to detect the generalized synchronization in unidirectionally coupled chaotic systems. As a consequence, one can predict (without recourse to an experimental auxiliary response system) whether an identical copy \(Y'\) of the response system connected to the driving system \(X\) will exhibit a behavior identical to the original response system \(Y\). In the domain of generalized synchronization, one can estimate the smoothness of the synchronization manifold. This estimate is based on a comparison of the global Lyapunov dimension with the dimension of the driving attractor.

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